Forcing and Large Cardinals

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1 Forcing basics

1.1 Background

A *partial order* is a pair $(\mathbb{P}, \leq)$ such that $\mathbb{P}$ is a set and $\leq$ is a transitive, reflexive binary relation on $\mathbb{P}$. We will often abuse notation and conflate the set $\mathbb{P}$ with the pair that puts structure on $\mathbb{P}$. All of our partial orders will have a maximal element, usually denoted by 1. Of course, we can always add a maximal element to any partial order.

If $\mathbb{P}$ is a partial order and $p, q \in \mathbb{P}$, we say p and q are *compatible* if there is $r \leq p, q$. We will abbreviate "p is incompatible with q" by $p \perp q$. An antichain is a set of pairwise-incompatible elements. A set $D \subseteq \mathbb{P}$ is said to be *dense* when for all $p \in \mathbb{P}$, there is $q \in D$ below p. A set is *open* when it is downward-closed. A set is *predense* if its downward closure is dense.

A subset F of a partial order is called a *filter* if it is upward-closed and directed: for all $p, q \in F$, there is $r \in F$ such that $r \leq p, q$. Suppose $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$. We say that a filter F is *$\mathcal{D}$-generic* if $D \cap F \neq \emptyset$ for all $D \in \mathcal{D}$. If M is a model of set theory and $\mathbb{P} \in M$, we say that a filter $G \subseteq \mathbb{P}$ is *generic over M* or *M-generic* if $G \cap D \neq \emptyset$ for every dense $D \in M$.

Exercise 1.1. *Suppose $\mathbb{P}$ is a partial order in a model of set theory M. Show that there is an M-generic filter $G \subseteq \mathbb{P}$ such that $G \in M$ iff there is $p \in \mathbb{P}$ such that every two elements below p are compatible.*

Exercise 1.2. *Show that if M is a model of set theory, $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ is M-generic, then G is a maximal filter.*

Exercise 1.3. *Suppose G is a filter over a partial order $\mathbb{P} \in M$. Show that the following are equivalent:*

1. *G meets every dense $D \in M$.*
2. *G meets every dense open $D \in M$.*
3. *G meets every predense $D \in M$.*
4. *G meets every maximal antichain $A \in M$.*

Working within a model of set theory V containing a partial order $\mathbb{P}$, we recursively construct the class of $\mathbb{P}$-names $V^{\mathbb{P}}$ by induction on rank. $V_0^{\mathbb{P}} = \{\emptyset\}$. Given $V_\alpha^{\mathbb{P}}$, we let $V_{\alpha+1}^{\mathbb{P}}$ be the set of all subsets of $\mathbb{P} \times V_\alpha^{\mathbb{P}}$. For limit λ, we let $V_\lambda^{\mathbb{P}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{P}}$.

Given a filter F over $\mathbb{P}$, which may or may not be a member of V, we recursively define the evaluation of $\mathbb{P}$-names by F. If τ is a $\mathbb{P}$-name, we let $\tau^F = \{\sigma^F : (p,\sigma) \in \tau \wedge p \in F\}$. We define $V[F]$ as the class of all evaluations of $\mathbb{P}$-names by F. An important subclass of $V^{\mathbb{P}}$ is the class of "check-names," which canonically evaluate to members of V under any filter. $\emptyset$ is a check-name. Given a set $x \in V$, we recursively define $\check{x} = \{(1, \check{y}) : y \in x\}$.

Exercise 1.4. *Show that for any filter F over a partial order $\mathbb{P} \in V$, and any $x \in V$, $\check{x}^F = x$.*

Another important name is the canonical name for the generic filter, $\dot{G} = \{(p, \check{p}) : p \in \mathbb{P}\}$.

If $\varphi(\tau_1, \dots, \tau_n)$ is a sentence in the language of set theory involving $\mathbb{P}$-names $\tau_1, \dots, \tau_n$, then given a filter G over $\mathbb{P}$, we can ask whether $V[G] \models \varphi(\tau_1^G, \dots, \tau_n^G)$. We say p forces φ over V, or $p \Vdash_{\mathbb{P}}^V \varphi$ when for every V-generic filter G with $p \in G$, we have $V[G] \models \varphi$.

For this course, we will take for granted the Forcing Theorem:

Theorem 1.5. *Suppose V is a model of ZFC and $\mathbb{P}$ is a partial order in V.*

1. *There is a computable transformation $\varphi(v_1, \dots, v_n) \mapsto \bar{\varphi}(v_1, \dots, v_n, v_{n+1}, v_{n+2})$ of the formulas in the language of set theory such that for any partial order $\mathbb{P} \in V$, any $p \in \mathbb{P}$, and any $\mathbb{P}$-names $\tau_1, \dots, \tau_n$,*

$$p \Vdash_{\mathbb{P}}^V \varphi(\tau_1, \dots, \tau_n) \Leftrightarrow V \models \bar{\varphi}(\tau_1, \dots, \tau_n, \mathbb{P}, p).$$

2. *If $G \subseteq \mathbb{P}$ is generic over V, and $V[G] \models \varphi(\tau_1^G, \dots, \tau_n^G)$, then there is $p \in G$ such that $p \Vdash_{\mathbb{P}}^V \varphi(\tau_1, \dots, \tau_n)$.*

3. *For all V-generic $G \subseteq \mathbb{P}$, $V[G] \models$ ZFC.*

The following is known as the maximality principle:

Lemma 1.6. *Suppose $\mathbb{P}$ is a partial order, $p \in \mathbb{P}$, and $p \Vdash \exists x \varphi(x)$. Then there is a $\mathbb{P}$-name τ such that $p \Vdash \varphi(\tau)$.*

Proof. The collection of $q \leq p$ such that there is some $\mathbb{P}$-name τ_q such that $q \Vdash \varphi(\tau_q)$ is dense. Let A be a maximal antichain below p of such q. Let

$$\tau = \{(r, \sigma) : \exists q \in A \exists r' (r \leq q, r' \text{ and } (r', \sigma) \in \tau_q\}.$$

If $G \subseteq \mathbb{P}$ is generic over V with $p \in G$, and let $q \in A \cap G$. Clearly $\tau^G \subseteq \tau_q^G$. If $x \in \tau_q^G$, then $x = \sigma^G$ for some σ such that $(r', \sigma) \in \tau_q$ and $r' \in G$, and we can find $r \leq r', q$ also in G, so $x \in \tau$. $\square$

We will sometimes need the fact that formulas of low complexity have a degree of absoluteness. $\Delta_0 \subseteq$ Form is the smallest collection of well-formed formulas in the language of set theory containing the atomic formulas, closed under propositional connectives, and closed under *bounded quantification*, which takes the form $(\forall x \in y)\varphi$ or $(\exists x \in y)\varphi$. Π_1 is the collection of formulas of the form $\forall x\varphi$, where φ is Δ_0, and Σ_1 is the collection of formulas of the form $\exists x\varphi$, where φ is Δ_0. Given the classes Σ_n and Π_n, Σ_{n+1} is the collection of formulas of the form $\exists x\varphi$, where φ is Π_n, and Π_{n+1} is the collection of formulas of the form $\forall x\varphi$, where φ is Σ_n. We will often say that a formula "is" Π_n of Σ_n when we really mean ZF proves that it is equivalent to a formula in this class. We let Δ_n be the class of formulas for which ZF proves they are equivalent to both a Σ_n and a Π_n formula.

Lemma 1.7. *Closure of the classes (mod ZF-provable equivalence):*

1. *For all n, $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$.*
2. *φ is Σ_n iff $\neg\varphi$ is Π_n.*
3. *Σ_n and Π_n are closed under conjunction, disjunction, and bounded quantification.*
4. *Σ_n is closed under existential quantification and Π_n is closed under universal quantification.*

Lemma 1.8. *Δ_0 formulas are absolute between transitive models. Σ_1 formulas are upwards-absolute and Π_1 formulas are downwards-absolute. Thus Δ_1 formulas are absolute between transitive models M, N of ZF such that $M \subseteq N$, and also absolute between possibly $\subseteq$-incomparable transitive models with respect to ordinal parameters.*

Proof. We sketch the proof of the final claim assuming the others. Suppose $\varphi(x)$ is Δ_1, and $\alpha \in \mathrm{Ord}^M \cap \mathrm{Ord}^N$. Then $M \models \varphi(\alpha) \Leftrightarrow L^M \models \varphi(\alpha) \Leftrightarrow L^N \models \varphi(\alpha) \Leftrightarrow N \models \varphi(\alpha)$. □

1.2 Separativity and Boolean Completions

A Boolean algebra is a structure that resembles the algebra of set operations on the powerset of a given set. It is a set $\mathbb{B}$ with distinguished top and bottom elements 1 and 0, binary operations $\wedge, \vee$, and a unary operation $\neg$ satisfying the following axioms:

1. (Commutativity) $p \vee q = q \vee p$ and $p \wedge q = q \wedge p$.
2. (Associativity) $p \vee (q \vee r) = (p \vee q) \vee r$ and $p \wedge (q \wedge r) = (p \wedge q) \wedge r$.
3. (Distributivity) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.
4. (Absorption) $p \wedge (p \vee q) = p$ and $p \vee (p \wedge q) = p$.

5. (Identity) $a \vee 0 = a$ and $a \wedge 1 = a$.

6. (Complementation) $p \vee (\neg p) = 1$ and $p \wedge (\neg p) = 0$.

We define a partial order on $\mathbb{B}$ by putting $p \leq q$ when $p \wedge q = p$. We use $p - q$ as an abbreviation for $p \wedge \neg q$.

Exercise 1.9. *Prove that the above definition of $\leq$ is indeed a partial order, with greatest element 1 and least elements 0. Show that $p \wedge q$ gives the greatest lower bound of p, q, and $p \vee q$ gives the least upper bound of p, q. Verify the De Morgan laws.*

Let us mention the following theorem, whose proof we will skip. It assures you that whatever equations you know hold for set algebras also hold for Boolean algebras generally.

Theorem 1.10 (Stone)**.** *Every Boolean algebra is isomorphic to a set algebra. Given a Boolean algebra $\mathbb{B}$, there is a set X and a family $\mathcal{A} \subseteq \mathcal{P}(X)$ containing X and closed under intersections, unions, and complements, such that under these standard operations it is isomorphic to $\mathbb{B}$.*

A Boolean algebra is called *complete* when every subset has a least upper bound. We aim to show that every partial order is "forcing-equivalent" to a complete Boolean algebra. The convenient thing about forcing with a complete Boolean algebra $\mathbb{B}$ is that for any statement φ in the forcing language of $\mathbb{B}$, there is a maximal element of $\mathbb{B}$ forcing this statement, denoted $||\varphi||$. This is also called the *Boolean value* of φ. It is simply the least upper bound of all conditions forcing φ.

Definition. *A partial order $(\mathbb{P}, \leq)$ is called* separative *if whenever $p \not\leq q$, there is $r \leq p$ such that $r \perp q$.*

Note that Boolean algebras are separative. Non-separative partial orders contain structure that is irrelevant for forcing purposes. We can get rid of it by taking a quotient. Given a partial order $\mathbb{P}$, we put $p \leq_s q$ when all $r \leq p$ are compatible with q. It is easy to check that $\leq_s$ is a partial order extending $\leq$. We put $p \sim_s q$ when $p \leq_s q$ and $q \leq_s p$. Since $\leq_s$ is transitive, $\sim_s$ is an equivalence relation. The set of equivalence classes is called the separative quotient $\mathbb{P}_s$.

Exercise 1.11. *Let $\mathbb{P}$ be any partial order.*

1. *Show that $\leq_s$ extends $\leq$ and that p, q are compatible in $(\mathbb{P}, \leq)$ iff $[p]_s, [q]_s$ are compatible in $(\mathbb{P}_s, \leq_s)$.*

2. *Show that $\mathbb{P}_s$ is separative.*

3. *Show that if $\mathbb{P}$ is separative, then $\mathbb{P}_s \cong \mathbb{P}$.*

Lemma 1.12. *Suppose $\mathbb{P}$ is a partial order in V. Let $[p]_s$ denote the equivalence class of $p \in \mathbb{P}$ in the separative quotient.*

1. *If $G \subseteq \mathbb{P}$ is generic over V, then $G_s := \{[p]_s : p \in G\}$ is a filter which is $\mathbb{P}_s$-generic over V.*

2. *If $G_s \subseteq \mathbb{P}_s$ is generic over V, then $G := \{p : [p]_s \in G_s\}$ is a filter which is $\mathbb{P}$-generic over V.*

Proof. Suppose $G \subseteq \mathbb{P}$ is generic over V. Let $D \in V$ be a dense open subset of $\mathbb{P}_s$. Let $D' = \{p : [p]_s \in D\}$. If $p \in \mathbb{P}$, find $q \leq_s p$ in D, and then find $r \leq q, p$. $[r]_s \in D$ since D is open and $\leq_s$ extends $\leq$. Thus D' is dense, so let $p \in G \cap D'$. Then $[p]_s \in D \cap G_s$. To verify G_s is is a filter, first note that directedness follows by the fact that $\leq_s$ extends $\leq$. For upwards closure, let $p \in G$ and suppose $p \leq_s q$. The set of $r \leq p$ that are below q is dense below p, so we must have $q \in G$ and thus $[q]_s \in G_s$.

Suppose $G_s \subseteq \mathbb{P}_s$ is generic over V. Let $D \in V$ be a dense subset of $\mathbb{P}$. Then $D' = \{[p]_s : p \in D\}$ is clearly dense in $\mathbb{P}_s$, so there is p such that $[p]_s \in G_s \cap D'$, and thus $p \in G \cap D$. We must verify that G is a filter. Upwards closure is easy. Let $p, q \in G$. Then the set E of r such that r is either below both p and q, or incompatible with one of them, is dense. So there is $r \in E$ such that $[r]_s \in G_s$. But incompatibility in $\mathbb{P}$ implies incompatibility in $\mathbb{P}_s$, so since G_s is a filter, this r must be below both p and q. □

Let $\mathbb{P}$ be a partial order. We will call an open subset $A \subseteq \mathbb{P}$ *regular* if whenever A is dense below p, then $p \in A$. Note that if $\mathbb{P}$ is separative, then the open set $U_p := \{q : q \leq p\}$ is regular. Given a set U, define:

$$\overline{U} = \{p : U \text{ is dense below } p\}.$$

Exercise 1.13. *Prove that the intersection of any family of regular open subsets of a partial order is regular and open.*

Exercise 1.14. *Let $\mathbb{P}$ be a partial order and let $U \subseteq \mathbb{P}$ be open. Show that $\overline{U}$ is the smallest regular open set containing U.*

Lemma 1.15. *Suppose $A, B \subseteq \mathbb{P}$ are open sets. Then:*

1. $\overline{A \cup \overline{B}} = \overline{A \cup B}$.

2. $\overline{A \cap \overline{B}} = \overline{A \cap B}$.

Proof. For (1), first note that $\overline{A \cup \overline{B}}$ is a regular open set containing the open set $A \cup B$, so it contains the minimal such set $\overline{A \cup B}$. Second, $\overline{A \cup B}$ is a regular open set containing B, so it contains $\overline{B}$, and since it also contains A, it contains $\overline{A \cup \overline{B}}$.

For (2), first note that $\overline{A \cap \overline{B}}$ is a regular open set containing $A \cap B$, so it contains $\overline{A \cap B}$. Second, suppose $p \in \overline{A \cap \overline{B}}$. Then $A \cap \overline{B}$ is dense below p, so each of $A, \overline{B}$ are separately dense below p. Since $\overline{B}$ is regular, $p \in \overline{B}$. By definition, B is dense below p. So both A and B are dense below p. Since they are open, $A \cap B$ is dense below p. Thus by definition, $p \in \overline{A \cap B}$. □

Theorem 1.16. *Let $\mathbb{P}$ be a separative partial order. There is a complete Boolean algebra $\mathcal{B}(\mathbb{P})$ that has a dense subset isomorphic to $\mathbb{P}$.*

Proof. Let $\mathbb{P}$ be as hypothesized. The universe of $\mathcal{B}(\mathbb{P})$ is the collection of all regular open subsets of $\mathbb{P}$. For regular open sets A, B, we define the operations as follows:

- $A \wedge B = A \cap B$.
- $A \vee B = \overline{A \cup B}$.
- $\neg A = \{p \in \mathbb{P} : U_p \cap A = \emptyset\}$.
- $1 = \mathbb{P}$, $0 = \emptyset$.

To verify the first four axioms for Boolean algebras, we use an induction argument along with Lemma 1.15, and the fact that these equations hold for the ordinary set operations. Suppose f is a function obtained by composing the operations $\wedge, \vee$ finitely many times. Let f' be the result of replacing each instance of $\vee$ with $\cup$. We suppose inductively that for a function f using $< n$ applications of $\wedge, \vee$, and any regular open sets $A_1, \ldots, A_n$ to plug in to the free variables of f, we have $f(A_1, \ldots, A_n) = \overline{f'(A_1, \ldots, A_n)}$.

Suppose then that f is a function using n applications of $\wedge, \vee$, and so $f = g_1 \wedge g_2$ or $g_1 \vee g_2$ for some functions g_1, g_2 with fewer applications. Suppose first that $f = g_1 \vee g_2$. Let $\vec{A}$ be a sequence of $n+1$ regular open sets. Then

$$f(\vec{A}) = \overline{g_1(\vec{A}) \cup g_2(\vec{A})}.$$

By induction, this equals

$$\overline{\overline{g_1'(\vec{A})} \cup \overline{g_2'(\vec{A})}}.$$

By two applications of Lemma 1.15, this equals

$$\overline{g_1'(\vec{A}) \cup g_2'(\vec{A})},$$

which is equal to $\overline{f'(\vec{A})}$ by the definition of $f \mapsto f'$. If $f = g_1 \wedge g_2$, then again by induction,

$$f(\vec{A}) = g_1(\vec{A}) \cap g_2(\vec{A}) = \overline{g_1'(\vec{A})} \cap \overline{g_2'(\vec{A})} = \overline{\overline{g_1'(\vec{A})} \cap \overline{g_2'(\vec{A})}}.$$

The last equality holds because the preceding term is already regular open. Applying Lemma 1.15 twice again, we get that this equals

$$\overline{g_1'(\vec{A}) \cap g_2'(\vec{A})} = \overline{f'(\vec{A})}.$$

This completes the induction argument. Thus the identities involving $\wedge$ and $\vee$ transfer from those for the corresponding set operations.

For the identity axiom, note that if A is regular open, then $\overline{A \cup \emptyset} = \overline{A} = A$, and $A \cap \mathbb{P} = A$.

For complements, we first check that if A is regular open, then $\neg A$ is regular open. It is open because if $U_p \cap A = \emptyset$ and $q \leq p$, then $U_q \cap A = \emptyset$. If $\neg A$ is dense below p, then suppose to the contrary that $p \notin \neg A$. Then $U_p \cap A \neq \emptyset$, so there is $q \leq p$ in A. But by assumption, there is $r \leq q$ such that $U_r \cap A = \emptyset$, a contradiction since A is open. For the complementation axiom, let A be regular open and let $p \in \mathbb{P}$. For all $q \leq p$, either $U_q \cap A = \emptyset$, or there is $r \leq q$ in A, so $A \cup \neg A$ is dense below p and thus $p \in \overline{A \cup \neg A}$. Also, there is no p which is in A and such that $U_p \cap A = \emptyset$.

To show completeness, we only need to check that every family of regular open sets has a greatest lower bound. But this follows from the fact that the intersection of such a family is regular open. Alternatively, suppose $\{A_i : i \in I\}$ is a collection of regular open sets. Then $\overline{\bigcup_{i \in I} A_i}$ is the smallest regular open set containing $\bigcup_{i \in I} A_i$, so it is smallest regular open set containing each A_i, and thus it is the least upper bound.

Finally, we note that if $\mathbb{P}$ is separative, then $p \mapsto U_p$ is a map into the regular open subsets of $\mathbb{P}$, with the property that $p \leq q$ iff $U_p \subseteq U_q$. Every regular nonempty regular open set A contains some U_p, so the range of the map is dense. $\square$

To complete our goal for this section, we just need to do the following:

Exercise 1.17. *Suppose $\mathbb{P}$ is a dense suborder of $\mathbb{Q}$. Show that if G is $\mathbb{P}$-generic over V, then the upward closure of G is a $\mathbb{Q}$-generic filter over V. Show that if G is $\mathbb{Q}$-generic over V, then $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over V.*

Exercise 1.18. *Suppose $\mathbb{A}, \mathbb{B}$ are complete Boolean algebras, $D \subseteq \mathbb{A}$ and $E \subseteq \mathbb{B}$ are dense, and $\pi : D \to E$ is an order-isomorphism. Then π can be extended to a Booelan isomorphism from $\mathbb{A}$ to $\mathbb{B}$.*

Exercise 1.19. *Suppose $\mathbb{B}$ is a complete Boolean algebra and $\{b_i : i \in I\} \subseteq \mathbb{B}$. Let $b = \sup_{i \in I} b_i$. Show that $a \wedge b = 0$ iff $a \wedge b_i = 0$ for all $i \in I$.*

1.3 Projections

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are partial orders. A map $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* when:

1. π is order-preserving.
2. $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$.
3. Whenever $p \leq \pi(q)$, there is $q' \leq q$ such that $\pi(q') \leq p$.

Exercise 1.20. *Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection. Show that if D is dense in $\mathbb{Q}$, then $\pi[D]$ is dense in $\mathbb{P}$, and if D is open and dense in $\mathbb{P}$, then $\pi^{-1}[D]$ is dense in $\mathbb{Q}$.*

A key example of a projection is the map from a product to one coordinate. Let $\mathbb{P}, \mathbb{Q}$ be partial orders. We define a partial order on $\mathbb{P} \times \mathbb{Q}$ by $(p_1, q_1) \leq$

(p_0, q_0) iff $p_1 \leq p_0$ and $q_1 \leq q_0$. We have two projection maps, $\pi_0 : \mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ and $\pi_1 : \mathbb{P} \times \mathbb{Q} \to \mathbb{Q}$, defined by $\pi_0(p, q) = p$ and $\pi_1(p, q) = q$.

More generally, consider a family of partial orders indexed by a set, $\langle \mathbb{P}_x : x \in X \rangle$. An *ideal* over X is a collection $I \subseteq \mathcal{P}(X)$ closed under pairwise unions and subsets. We can form the I-support product of the $\mathbb{P}_x$'s:

$$\prod_{x \in X}^{I} \mathbb{P}_x := \{f \in \prod_{x \in X} \mathbb{P}_x : \{x : f(x) \neq 1_{\mathbb{P}_x} \in I\}\}.$$

We put a partial order on $\prod_x^I \mathbb{P}_x$ by putting $f \leq g$ when $f(x) \leq g(x)$ for all $x \in X$. Typical examples include taking I to be all finite subsets of X, or all countable subsets of X, or simply all subsets of X. These are known respectively as finite-support, countable-support, and full-support products.

If $Y \subseteq X$, then $I \upharpoonright Y := \{A \in I : A \subseteq Y\}$ is an ideal on Y. We have a natural projection

$$\pi : \prod_{x \in X}^{I} \mathbb{P}_x \to \prod_{x \in Y}^{I \upharpoonright Y} \mathbb{P}_x$$

given by $\pi(f) = f \upharpoonright Y$. Note that for each such Y,

$$\prod\nolimits_{x \in X}^{I} \mathbb{P}_x \cong \prod_{x \in Y}^{I \upharpoonright Y} \mathbb{P}_x \times \prod_{x \in X \setminus Y}^{I \upharpoonright (X \setminus Y)} \mathbb{P}_x.$$

Projections are a very general tool for doing iterations of forcing. If $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection and G is a filter over $\mathbb{P}$, then we often write $\mathbb{Q}/G$ for $\pi^{-1}[G]$. Note that this is usually not separative.

Theorem 1.21. *Suppose $\mathbb{P}, \mathbb{Q}$ are partial orders in V and $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection.*

1. *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}/G$-generic over $V[G]$, then H is $\mathbb{Q}$-generic over V, and G is the upward-closure of $\pi[H]$.*

2. *If H is $\mathbb{Q}$-generic over V and G is the upward-closure of $\pi[H]$, then G is $\mathbb{P}$-generic over V, and H is $\mathbb{Q}/G$-generic over $V[G]$.*

Proof. For (1), suppose G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}/G$-generic over $V[G]$. Let $D \in V$ be a dense open subset of $\mathbb{Q}$. Let $D' \in V[G]$ be the set $D \cap \pi^{-1}[G]$. We claim D' is dense in $\mathbb{Q}/G$. This suffices, since then $H \cap D \neq \emptyset$. So let $q \in \mathbb{Q}/G$ and let $D_q = D \cap U_q$, which is dense below q. $\pi[D_q]$ is dense below $\pi(q)$ so by genericity there is $p \in \pi[D_q] \cap G$. $p = \pi(q')$ for some $q' \leq q$, and $q' \in D'$. To show the last claim, note for any $p \in G$, $\{q : \pi(q) \leq p\}$ is dense in $\mathbb{Q}/G$.

For (2), suppose H is $\mathbb{Q}$-generic over V, and let G be the upward-closure of $\pi[H]$ in $\mathbb{P}$. Let $D \in V$ be a dense open subset of $\mathbb{P}$. Then $\pi^{-1}[D]$ is dense in $\mathbb{Q}$, so $H \cap \pi^{-1}[D] \neq \emptyset$, and G is $\mathbb{P}$-generic over V. Now suppose $D \in V[G]$ is a dense open subset of $\mathbb{Q}/G$. Let $\dot{D} \in V$ be a $\mathbb{P}$-name such that $\dot{D}^G = D$. Let $p_0 \in G$ force that $\dot{D}$ is a dense open subset of $\mathbb{Q}/\dot{G}$, and let $q_0 \in H$ be such that $\pi(q_0) = p_0$. We claim that the set $E = \{q \leq q_0 : \pi(q) \Vdash q \in \dot{D}\}$ is dense below q_0. Let $q \leq q_0$. Let $p \leq \pi(q)$ be such that for some $d \leq q$, $p \Vdash \check{d} \in \dot{D}$. Since p forces $\pi(d) \in \dot{G}$, let $p' \leq p, \pi(d)$. Let $d' \leq d$ be such that $\pi(d') \leq p'$. Then $\pi(d') \Vdash \check{d}' \in \mathbb{Q}/\dot{G}$, and since $\dot{D}$ is forced to be open, $\pi(d') \Vdash \check{d}' \in \dot{D}$. Thus E is dense below q_0, so let $q_1 \in E \cap H$. Then $q_1 \in D \cap H$, showing that H is $\mathbb{Q}/G$-generic over $V[G]$. □

Exercise 1.22. *Suppose $\pi : \mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ is the natural projection and G is $\mathbb{P}$-generic over V. Show that the separative quotient of $(\mathbb{P} \times \mathbb{Q})/G$ is isomorphic to the separative quotient of $\mathbb{Q}$.*

As a corollary of Theorem 1.21 and the above exercise, we have:

Corollary 1.23. *For partial orders $\mathbb{P}, \mathbb{Q} \in V$, the following are equivalent:*

1. *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V.*
2. *G is $\mathbb{P}$-generic over V, and H is $\mathbb{Q}$-generic over $V[G]$.*
3. *H is $\mathbb{Q}$-generic over V, and G is $\mathbb{P}$-generic over $V[H]$.*

1.4 Complete Embeddings and Forcing Equivalence

Suppose $\mathbb{P}, \mathbb{Q}$ are complete Boolean algebras. A map $e : \mathbb{P} \to \mathbb{Q}$ is called an *embedding* if is an injection that preserves the algebraic operations. It is called a *complete embedding* if it preserves least upper bounds. $\mathbb{P}$ is called a *complete subalgebra* of $\mathbb{Q}$ if it is a Boolean substructure and the identity map is a complete embedding.

Exercise 1.24. *Suppose $e : \mathbb{P} \to \mathbb{Q}$ is an embedding. Show that the following are equivalent:*

1. *e is complete.*
2. *For all maximal antichains $A \subseteq \mathbb{P}$, $e[A]$ is maximal in $\mathbb{Q}$.*
3. *For all $q \in \mathbb{Q}$, there is $p \in \mathbb{P}$ such that for all $p' \leq p$, $e(p') \wedge q \neq 0$.*
4. *For all $\mathbb{Q}$-generic H, $e^{-1}[H]$ is $\mathbb{P}$-generic.*

Remark 1.25. *Conditions (2)–(4) above also make sense for partial orders generally. We sometimes speak of complete embeddings in this general context, by which we mean one of these conditions.*

Suppose $\mathbb{P} \subseteq \mathbb{Q}$ are Boolean algebras and F is a filter on $\mathbb{P}$. Then the upward closure of F is a filter on $\mathbb{Q}$. Let I be its dual ideal. We can form the quotient Boolean algebra $\mathbb{Q}/I$ by putting $q \sim q'$ iff the symmetric difference $(q - q') \vee (q' - q) \in I$, and defining the operations on equivalence classes as the equivalence class of the operations applied to representatives. It is straightforward to check that this is well-defined.

Lemma 1.26. *Suppose $\mathbb{P}, \mathbb{Q}$ are complete Boolean algebras in V and $e : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. If $G \subseteq \mathbb{P}$ is generic over V, then let $\mathbb{Q}/G$ denote the quotient algebra of $\mathbb{Q}$ by the dual ideal to the upward closure of $e[G]$. Then $\mathbb{Q}/G$ is complete in $V[G]$.*

Proof. Without loss of generality, $\mathbb{P} \subseteq \mathbb{Q}$ and e is the identity map. Suppose $p \Vdash \dot{S} \subseteq \mathbb{Q}/G$. Let $a = \sup\{q \wedge ||[q]_{\dot{G}} \in \dot{S}|| : q \in \mathbb{Q}\}$. Let G be generic with $p \in G$. If $[q]_G \in \dot{S}^G$, then $[||[q]_{\dot{G}} \in \dot{S}||]_G = [1]_G$ so $[q]_G = [q \wedge ||[q]_{\dot{G}} \in \dot{S}||]_G$ and thus $[q]_G \leq [a]_G$.

We want to show that $[a]_G$ is the least upper bound of $\dot{S}^G$. Suppose $[0]_G < [b]_G \leq [a]_G$. It suffices to find some q such that $[b \wedge q \wedge ||[q]_{\dot{G}} \in \dot{S}||]_G \neq [0]_G$, since such a q must be in $\dot{S}^G$. If such a q does not exist, then let $p' \leq p$ force this and also force that $[0]_G < [b \wedge a]_G$. We must have that

$$p' \wedge b \wedge q \wedge ||[q]_{\dot{G}} \in \dot{S}|| = 0 \text{ for all } q \in \mathbb{Q}.$$

For otherwise we could take a $\mathbb{Q}$-generic H containing $p' \wedge b \wedge q \wedge ||[q]_{\dot{G}} \in \dot{S}||$ for some q, and then if $G' = H \cap \mathbb{P}$, we would have $[b \wedge q \wedge ||[q]_{\dot{G}} \in \dot{S}||]_{G'} \neq [0]_{G'}$. Taking the supremum in $\mathbb{Q}$, we have $p' \wedge b \wedge a = 0$, and so $p' \Vdash [b \wedge a]_{\dot{G}} = [0]_{\dot{G}}$, a contradiction. □

Lemma 1.27. *Suppose $\mathbb{P}, \mathbb{Q}$ are complete Boolean algebras, and $\mathbb{P}$ is a complete subalgebra of $\mathbb{Q}$. Then the map $\pi : \mathbb{Q} \to \mathbb{P}$ given by:*

$$\pi(q) = \inf\{p \in \mathbb{P} : q \leq p\}$$

is a projection. Furthermore, $\pi(q) = ||[\check{q}]_{\dot{G}} > 0||$.

Proof. It is clear that π preserves order and that $\pi(1) = 1$. Suppose $p \leq \pi(q)$ is in $\mathbb{P} \setminus \{0\}$. Then $p \wedge q > 0$, since otherwise $q \leq \neg p$, and then $p \leq \pi(q) \leq \neg p$, which is impossible. Thus $0 < p \wedge q \leq q$, and $\pi(p \wedge q) \leq p$.

This also shows that no $p \leq \pi(q)$ can force $[\check{q}]_{\dot{G}} = 0$, so $\pi(q) \leq ||[\check{q}]_{\dot{G}} > 0||$. Further, since $q \leq \pi(q)$, $\neg\pi(q) \leq \neg q$, so any nonzero $p \leq \neg\pi(q)$ forces $[\check{q}]_{\dot{G}} = 0$. □

Lemma 1.28. *Suppose $\mathbb{P}, \mathbb{Q}$ are complete Boolean algebras in V, and $\mathbb{P}$ is a complete subalgebra of $\mathbb{Q}$.*

1. *If G is $\mathbb{P}$-generic over V and $\tilde{H}$ is $\mathbb{Q}/G$-generic over $V[G]$, then $H := \{q : [q]_G \in \tilde{H}\}$ is $\mathbb{Q}$-generic over V.*
2. *If H is $\mathbb{Q}$-generic over V, then $G = H \cap \mathbb{P}$ is $\mathbb{P}$-generic over V, and $\tilde{H} := \{[q]_G : q \in H\}$ is $\mathbb{Q}/G$-generic over $V[G]$.*

Proof. For (1), suppose $G, \tilde{H}$ are as hypothesized, and let $D \in V$ be a dense subset of $\mathbb{Q}$. It suffices to show that $\tilde{D} := \{[q]_G : q \in D\}$ is dense in $\mathbb{Q}/G$. If not, let $p \in G$ and $q \in \mathbb{Q}$ be such that $p \Vdash [\check{q}]_{\dot{G}} > 0 \wedge \neg\exists d \in \check{D}([d]_{\dot{G}} \leq [\check{q}]_{\dot{G}})$. Then $p \wedge q \neq 0$, so let $d \in D$ be such that $d \leq p \wedge q$. Let H' be $\mathbb{Q}$-generic over V with

$d \in H$. Then $G' = H' \cap \mathbb{P}$ is $\mathbb{P}$-generic over V, $p \in G'$, and $0 < [d]_{G'} \leq [q]_{G'}$, a contradiction.

For (2), let H be as hypothesized, and let $G = H \cap \mathbb{P}$. Suppose D is a dense open subset of $\mathbb{Q}/G$ in $V[G]$. Let $\dot{D}$ be a $\mathbb{P}$-name such that $\dot{D}^G = D$. We may assume that $\dot{D}$ is forced to be open and dense. Let $\pi : \mathbb{Q} \to \mathbb{P}$ be the projection of Lemma 1.27. Let $E = \{q \in \mathbb{Q} : \pi(q) \Vdash [\check{q}]_{\dot{G}} \in \dot{D}\}$. We claim E is dense. Since $\pi(q) \Vdash [\check{q}]_{\dot{G}} > 0$, there is some $p \leq \pi(q)$ and some q' such that $p \Vdash [q']_{\dot{G}} \leq [q]_{\dot{G}}$ and $[q']_{\dot{G}} \in \dot{D}$. We must have $p \perp (q' - q)$, so if $q'' = q' \wedge q$, then $p \Vdash [q'']_{\dot{G}} \in \dot{D}$ and $p \wedge q'' > 0$. Thus $p \wedge q'' \in E$, and $p \wedge q'' \leq q$. Thus there is $q \in H \cap E$, and $\pi(q) \in G$, so $[q]_G \in D$. □

Lemma 1.29. *Suppose $\mathbb{P}$ is a complete subalgebra of $\mathbb{Q}$, and $\pi : \mathbb{Q} \to \mathbb{P}$ is the projection of Lemma 1.27. Let $G \subseteq \mathbb{P}$ be generic. Then the π-based quotient forcing $\pi^{-1}[G]$ is forcing-equivalent to the quotient boolean algebra $\mathbb{Q}/G$.*

Proof. Since $\pi(q) = ||[q] \neq 0||$, $\pi^{-1}[G]$ is exactly those elements whose equivalence class is nonzero in $\mathbb{Q}/G$. The separative quotient of $\pi^{-1}[G]$ is simply equal to $\mathbb{Q}/G \setminus \{[0]\}$. For if $q_0, q_1 \in \pi^{-1}[G]$, and $(q_0 - q_1) \vee (q_1 - q_0)$ is in the dual ideal to G, then for every $q_2 \in \pi^{-1}[G]$ such that $q_2 \leq q_1$, $q_2 \wedge q_0 \wedge q_1$ is positive in $\mathbb{Q}/G$. So being equivalent in the sense of the dual ideal to G implies being equivalent in the separative quotient. If q_0, q_1 are not equivalent modulo G, then one of $(q_0 - q_1), (q_1 - q_0)$ is positive, and this separates q_0 from q_1. □

Theorem 1.30. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are partial orders. $\mathcal{B}(\mathbb{P}_s) \cong \mathcal{B}(\mathbb{Q}_s)$ if and only if the following holds. Letting $\dot{G}, \dot{H}$ be the canonical names for the generic filters for $\mathbb{P}, \mathbb{Q}$ respectively, there is a $\mathbb{P}$-name $\dot{h}$ and a $\mathbb{Q}$-name $\dot{g}$ such that:*

1. *$\Vdash_{\mathbb{P}} \dot{h}$ is a $\mathbb{Q}$-generic filter,*
2. *$\Vdash_{\mathbb{Q}} \dot{g}$ is a $\mathbb{P}$-generic filter,*
3. *$\Vdash_{\mathbb{P}} \dot{G} = \dot{g}^{\dot{h}}$, and $\Vdash_{\mathbb{Q}} \dot{H} = \dot{h}^{\dot{g}}$.*

Proof. By Lemma 1.12 and Theorem 1.16, for any partial order $\mathbb{R}$, we can easily translate between $\mathbb{R}$-names and $\mathcal{B}(\mathbb{R}_s)$-names, so it suffices to assume $\mathbb{P}$ and $\mathbb{Q}$ are complete Boolean algebras. For the forward direction, if $\iota : \mathbb{P} \to \mathbb{Q}$ is an isomorphism, it works to set $\dot{h}$ to be the $\mathbb{P}$-name for $\iota[\dot{G}]$, and $\dot{g}$ to be the $\mathbb{Q}$-name for $\iota^{-1}[\dot{H}]$.

Suppose the names $\dot{g}, \dot{h}$ satisfy the above hypotheses. Let $e : \mathbb{P} \to \mathbb{Q}$ be defined by $e(p) = ||\check{p} \in \dot{g}||$. It is easy to see that e preserves meets. It also preserves joins because for generic filters G, $a \vee b \in G$ iff $a \in G$ or $b \in G$. It is also immediate that it is order- and incompatibility-preserving.

Claim 1: e preserves maximal antichains. This is because if $A \subseteq \mathbb{P}$ is a maximal antichain, then it is forced by $\mathbb{Q}$ that $\dot{g} \cap A \neq \emptyset$, so it is forced that $e[A] \cap \dot{H} \neq \emptyset$. This cannot happen if there is $q > 0$ such that $q \perp e[A]$.

Claim 2: $e(p) > 0$ for all $p > 0$. Suppose $p > 0$ is in $\mathbb{P}$. Let G be $\mathbb{P}$-generic over V with $p \in G$. Let $h = \dot{h}^G$, which is $\mathbb{Q}$-generic over V. Let $g = \dot{g}^h$. Then $p \in G = g$, so some condition in $\mathbb{Q}$ forces $p \in \dot{g}$.

Claim 3: The range of e is dense. Suppose $q > 0$ is in $\mathbb{Q}$. Let H_0 be $\mathbb{Q}$-generic over V with $q \in H_0$. Let $g = \dot{g}^{H_0}$. By hypothesis, $H_0 = \dot{h}^g$. Since g is generic, there is $p \in g$ such that $p \Vdash q \in \dot{h}$. Let H be any other $\mathbb{Q}$-generic filter with $e(p) \in H$. Then $p \in \dot{g}^H$ and $q \in \dot{h}^g = H$. By separativity, $e(p) \leq q$.

It follows from Claim 2 that e is injective. By Claim 1, e is a complete embedding. Since its range is dense, if $q \in \mathbb{Q}$, then $q = \sup\{e(p) : e(p) \leq q\} = e(\sup\{p : e(p) \leq q\})$, so e is surjective. □

Exercise 1.31. *Suppose $\mathbb{P}, \mathbb{Q}$ are partial orders. Show that the following are equivalent:*

1. *There is a complete embedding $e : \mathcal{B}(\mathbb{P}_s) \to \mathcal{B}(\mathbb{Q}_s)$.*
2. *There is a projection $\pi : \mathbb{Q}_s \to \mathcal{B}(\mathbb{P}_s)$.*
3. *There is a $\mathbb{Q}$-name $\dot{g}$ for a $\mathbb{P}$-generic filter such that for all $p \in \mathbb{P}$, there is $q \in \mathbb{Q}$ such that $q \Vdash p \in \dot{g}$.*

1.5 Iterations

Suppose $\mathbb{P}$ is a partial order and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a partial order. We define $\mathbb{P} * \dot{\mathbb{Q}}$ as the set of pairs $(p, \dot{q})$ such that $p \in \mathbb{P}$ and $1 \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$. But this is a bit problematic because of the following example. Let τ be a $\mathbb{P}$-name and let α be an ordinal. Suppose $(p, \sigma) \in \tau$, and $q \perp p$. Define:

$$\sigma' = \{(r, x) : r \leq p \text{ and } r \Vdash x \in \sigma\} \cup \{(q, \check{\alpha})\}.$$

Let $\tau' = \tau \cup \{(p, \sigma')\}$. Then $1 \Vdash \tau' = \tau$. So there is typically a proper class of $\mathbb{P}$-names that are forced to be elements of some other $\mathbb{P}$-name.

But whenever $\tau \in V$ is a $\mathbb{P}$-name, G is $\mathbb{P}$-generic over V, $x \in \tau^G$, then there is $(p, \sigma) \in \tau$ such that $p \in G$ and $\sigma^G = x$. If $(\sigma')^G = \sigma^G$, there is $q \in G$ such that $q \Vdash \sigma' = \sigma$. So if $1 \Vdash \sigma \in \tau$, then we can select a maximal antichain $A \subseteq \mathbb{P}$ of conditions q such that for some $\sigma_q \in \operatorname{trcl}(\tau)$, $q \Vdash \sigma_q = \sigma$. We can fuse these together by taking:

$$\sigma' = \{(r, x) : \exists q \in A(r \leq q \text{ and } r \Vdash x \in \sigma_q\}.$$

Then $1 \Vdash \sigma' = \sigma$, and $\operatorname{rank}(\sigma') \leq \max\{\operatorname{rank}(\mathbb{P}), \operatorname{rank}(\tau)\}$. Thus we have a *set* that functions as a complete collection of representatives of names for elements of τ.

Thus officially, we can define $\mathbb{P} * \dot{\mathbb{Q}}$ as the collection of $(p, \dot{q})$ such that $p \in \mathbb{P}$, $\dot{q} \in V_{\max\{\operatorname{rank}(\mathbb{P}), \operatorname{rank}(\dot{\mathbb{Q}})\}}$, and $1 \Vdash \dot{q} \in \dot{\mathbb{Q}}$. We say $(p_1, \dot{q}_1) \leq (p_0, \dot{q}_0)$ iff $p_1 \leq p_0$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_0$. It is easy to see that this order is transitive, but it may not be antisymmetric: There will often be distinct names $\dot{q}_0, \dot{q}_1$ such that for some p, $p \Vdash \dot{q}_0 = \dot{q}_1$. So to turn this from a preorder into a partial order, we mod out by the equivalence relation $(p, \dot{q}_0) \sim (p, \dot{q}_1)$ when $p \Vdash \dot{q}_0 = \dot{q}_1$. So really officially, we take $\mathbb{P} * \dot{\mathbb{Q}}$ to be the induced partial order on such equivalence classes. Note that this equivalence is finer than the separativity equivalence.

Note that there is a natural projection $\pi : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P}$ given by $\pi(p, \dot{q}) = p$. Note also that the map $p \mapsto (p, \dot{1})$ is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \dot{\mathbb{Q}}$. We have two notions of quotient forcing, one defined via projections, and the other via complete embeddings. As we have seen, they yield equivalent notions of forcing after forcing with $\mathbb{P}$. We now show that these are equivalent to forcing with the evaluation of $\dot{\mathbb{Q}}$.

Lemma 1.32. *Suppose $\mathbb{P}$ is a poset, $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, and $G \subseteq \mathbb{P}$ is generic. Then $(\mathbb{P} * \dot{\mathbb{Q}})/G$ and $\dot{\mathbb{Q}}^G$ have isomorphic separative quotients.*

Proof. Let $G \subseteq \mathbb{P}$ be generic. Let us take $(\mathbb{P} * \dot{\mathbb{Q}})/G$ in the projection sense. First note that $(\mathbb{P} * \dot{\mathbb{Q}})/G = G \times \{\dot{q} : (1, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}\}$ and $\dot{\mathbb{Q}}^G = \{\dot{q}^G : (1, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}\}$. For each $\mathbb{P}$-name for an element $\dot{q}$ of $\dot{\mathbb{Q}}$, the equivalence class of $(1, \dot{q})$ in the separative quotient includes $G \times \{\dot{q}\}$. For let $p \in G$ be arbitrary. For every $(p', \dot{q}') \leq (p, \dot{q})$, we have $(p', \dot{q}') \leq (1, \dot{q})$, and for every $(p', \dot{q}') \leq (1, \dot{q})$, there is $p'' \in G$ such that $(p'', \dot{q}') \leq (p, \dot{q})$.

Claim: For $\dot{q}_0^G, \dot{q}_1^G \in \dot{\mathbb{Q}}^G$, $\dot{q}_1^G \leq_s \dot{q}_0^G$ iff $(1, \dot{q}_1) \leq_s (1, \dot{q}_0)$.

Given the claim, we define a map $\phi : \dot{\mathbb{Q}}_s^G \to ((\mathbb{P} * \dot{\mathbb{Q}})/G)_s$ by $\phi([\dot{q}^G]) = [(1, \dot{q})]$, where the square brackets indicate the equivalence class. The claim implies that the map is well-defined on the equivalence classes, and order-preserving and injective. By the argument in the first paragraph, it is surjective.

Suppose $\dot{q}_1^G \leq_s \dot{q}_0^G$. Let $p_1 \in G$ be such that $p_1 \Vdash \dot{q}_1 \leq_s \dot{q}_0$. Let $(p_2, \dot{q}_2) \leq (1, \dot{q}_1)$ be in $(\mathbb{P} * \dot{\mathbb{Q}})/G$. Let $(p_3, \dot{q}_3)$ be such that $p_3 \in G$, $p_3 \leq p_1, p_2$, and $p_3 \Vdash \dot{q}_3 \leq \dot{q}_2, \dot{q}_0$. Thus $(p_2, \dot{q}_2)$ is compatible with $(1, \dot{q}_0)$, so $(1, \dot{q}_1) \leq_s (1, \dot{q}_0)$.

Suppose $\dot{q}_1^G \not\leq_s \dot{q}_0^G$. Let $(p, \dot{q}_2)$ be such that $p \in G$ and $p \Vdash \dot{q}_2 \leq \dot{q}_1$ and $\dot{q}_2 \perp \dot{q}_0$. Then $(p, \dot{q}_2) \leq (1, \dot{q}_1)$ and $(p, \dot{q}_2) \perp (1, \dot{q}_0)$. Thus $(1, \dot{q}_1) \not\leq_s (1, \dot{q}_0)$. □

Corollary 1.33. *Suppose $\mathbb{P}$ is a poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset. Then forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as forcing with $\mathbb{P}$ and then with $\dot{\mathbb{Q}}^G$, where G is the generic for $\mathbb{P}$. Furthermore, if $G \subseteq \mathbb{P}$ is generic, then in $V[G]$, $\dot{\mathbb{Q}}^G$ is forcing-equivalent to $(\mathbb{P} * \dot{\mathbb{Q}})/G$ (in either sense).*

If $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration and $\dot{\mathbb{R}}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a partial order, then we can form $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$. This can also be written as $\mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}})$, via a translation of names. Every $\mathbb{P} * \dot{\mathbb{Q}}$-name τ can be rewritten as a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name $\tilde{\tau}$ and vice versa. This is defined inductively on rank. Given τ, we put

$$\tilde{\tau} = \{(p, \langle \dot{q}, \tilde{\sigma} \rangle) : ((p, \dot{q}), \sigma) \in \tau\}.$$

(The angled brackets indicate that we take a *name* for the given ordered pair.) A straightforward induction shows that whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic, then $\tau^{G*H} = (\tilde{\tau}^G)^H$, and for any $\mathbb{P}*\dot{\mathbb{Q}}$-names $\tau_1, \ldots, \tau_n$ and any formula $\varphi(v_1, \ldots, v_n)$,

$$(p, \dot{q}) \Vdash^V_{\mathbb{P}*\dot{\mathbb{Q}}} \varphi(\tau_1, \ldots, \tau_n) \Leftrightarrow p \Vdash^V_{\mathbb{P}} (\dot{q} \Vdash^{V[\dot{G}]}_{\dot{\mathbb{Q}}} \varphi(\tilde{\tau}_1, \ldots, \tilde{\tau}_n)).$$

This allows a translation of $\dot{\mathbb{R}}$ and of names for its elements, in a way that respects the ordering in the iterations.

We will revisit iterations and introduce those of transfinite length in later chapters.

2 Easton's Theorem

Easton's Theorem says that cardinal arithmetic on regular cardinals can be whatever we want, subject to some basic constraints. Let us first describe those constraints.

Proposition 2.1. *If μ is a cardinal and $\kappa \leq \lambda$ are cardinals, then $\mu^\kappa \leq \mu^\lambda$.*

Theorem 2.2 (König)**.** *Suppose $\langle \kappa_i : i \in I \rangle$ and $\langle \lambda_i : i \in I \rangle$ are sequences of cardinals such that $\kappa_i < \lambda_i$ for all i. Then $\sum_i \kappa_i < \prod_i \lambda_i$.*

Proof. S be the disjoint union of the κ_i and let $f : S \to \prod_i \lambda_i$ be a function. It suffices to show that f is not surjective. For each i, there is $\alpha_i < \lambda_i$ such that for all $\beta < \kappa_i$, $f(\beta)(i) \neq \alpha_i$. Then the sequence $\langle \alpha_i : i \in I \rangle$ is not in $f[S]$. □

Corollary 2.3. *For all cardinals $\lambda \geq 2$ and all infinite cardinals κ, $\text{cf}(\lambda^\kappa) > \kappa$.*

Proof. Let $\langle \kappa_i : i < \kappa \rangle$ be such that each $\kappa_i < \lambda^\kappa$, and let $\lambda_i = \lambda^\kappa$ for each i. By König's Theorem, $\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \lambda_i = (\lambda^\kappa)^\kappa = \lambda^\kappa$. □

Theorem 2.4 (Easton)**.** *Assume GCH. Let F be a function such that $\text{dom}\, F$ is contained in the regular cardinals, $\text{ran}\, F$ is contained in the cardinals, and*

1. *For $\kappa_0 < \kappa_1$ in $\text{dom}\, F$, $F(\kappa_0) \leq F(\kappa_1)$.*
2. *For all $\kappa \in \text{dom}\, F$, $\text{cf}(F(\kappa)) > \kappa$.*

Then there is a forcing extension preserving all regular cardinals, in which for all $\kappa \in \text{dom}\, F$, $2^\kappa = F(\kappa)$.

Exercise 2.5. *Suppose that $V \subseteq W$ are transitive models of set theory and κ is a regular cardinal in both models. Show that for all ordinals $\alpha \in V$, if $V \models \text{cf}(\alpha) = \kappa$, then $W \models \text{cf}(\alpha) = \kappa$.*

We will prove the above theorem assuming that F is a *set* rather than a proper class. If we start with a model containing an inaccessible cardinal κ, then any function F as above with domain contained in κ will preserve the inaccessibility of κ. In the extension, V_κ will be a model of ZFC plus any cardinal arithmetic on the regular cardinals below κ satisfying the above constraints.

However, we don't need inaccessible cardinals to achieve the consistency of, for example, ZFC + "For all regular κ, $2^\kappa = \kappa^{++}$." This can be done assuming just the consistency of ZFC, using a forcing that is a proper class. Class forcing involves some technical issues that we do not plan to treat in this course. For a definitive reference on class forcing, see the book by Prof. Sy Friedman.

Suppose κ is a regular cardinal and λ is an ordinal. We define a partial order $\text{Add}(\kappa, \lambda)$, which is otherwise known as "adding λ Cohen subsets of κ." We take the set of all partial functions on $\kappa \times \lambda$ into $\{0, 1\}$ of size $< \kappa$ ordered by $p \leq q$ if $p \supseteq q$. Easton's Theorem is that if V satisfies GCH, then the desired model is obtained by forcing with the "Easton-support" product of $\text{Add}(\kappa, F(\kappa))$ over $\kappa \in \text{dom}\, F$.

2.1 Some combinatorial forcing lemmas

A partial order is called *κ-closed* if every descending sequence of length less than κ has a lower bound. A partial order is called *κ-distributive* if the intersection of fewer than κ dense open sets is dense.

Lemma 2.6. *If $\mathbb{P}$ is κ-closed, then $\mathbb{P}$ is κ-distributive.*

Proof. Let $\mu < \kappa$ and let $\langle D_i : i < \mu \rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Let $p_0 \in \mathbb{P}$ be arbitrary. We will construct a descending sequence $\langle p_i : i \leq \mu \rangle$ such that $p_{i+1} \in D_i$. Since the sets are open, $p_\mu \in \bigcap_{i<\mu} D_i$. Given p_i, let $p_{i+1} \in D_i$ be less than p_i. Given the sequence $\langle p_i : i < \alpha \rangle$ up to a limit ordinal α, use κ-closure to find a lower bound p_α. $\square$

Lemma 2.7. *Suppose $\mathbb{P}$ is a separative partial order. $\mathbb{P}$ is κ-distributive iff forcing with $\mathbb{P}$ adds no new sequences of ordinals of length $< \kappa$.*

Proof. Suppose $\mathbb{P}$ is κ-distributive. Let $\mu < \kappa$ and let $\dot{f}$ be a $\mathbb{P}$-name for a function from μ to ordinals. For each $\alpha < \mu$, the set D_α of p that decide $\dot{f}(\alpha)$ is dense open. If $p \in \bigcap_{\alpha<\mu} D_\alpha$, then p decides all values of $\dot{f}$.

Suppose $\mu < \kappa$ and $\mathbb{P}$ adds no new sequences of ordinals of length μ. Let $\langle D_\alpha : \alpha < \mu \rangle$ be a sequence of dense open subsets of $\mathbb{P}$. For each α, let $A_\alpha \subseteq D_\alpha$ be a maximal antichain. If $G \subseteq \mathbb{P}$ is generic, then for each α there is a unique $p_\alpha \in A_\alpha \cap G$. The sequence $\langle p_\alpha : \alpha < \mu \rangle$ is in V. Let q be such that $q \Vdash \forall \alpha < \mu (\dot{G} \cap \check{A}_\alpha = \{\check{p}_\alpha\})$. By separativity, $q \leq p_\alpha$ for all $\alpha < \mu$, so $q \in \bigcap_{\alpha<\mu} D_\alpha$ by openness. $\square$

A partial order $\mathbb{P}$ is said to have the *κ-chain-condition (κ-c.c.)* if all antichains $A \subseteq \mathbb{P}$ have cardinality $< \kappa$.

Exercise 2.8. *Show that $\mathbb{P}$ is κ-c.c. iff $\mathcal{B}(\mathbb{P})$ has no* chains *of length κ.*

Lemma 2.9. *Suppose κ is a regular cardinal and $\mathbb{P}$ is κ-c.c. Then $\mathbb{P}$ preserves regular cardinals $\geq \kappa$.*

Proof. Suppose λ is a regular cardinal $\geq \kappa$, and let $\dot{f}$ be a $\mathbb{P}$-name for a function from δ to λ, where $\delta < \lambda$. For each $\alpha < \delta$, let $A_\alpha \subseteq \mathbb{P}$ be a maximal antichain of conditions deciding $\dot{f}(\alpha)$. Each A_α has size $< \kappa$. For each α, let

$$X_\alpha = \{\beta : \exists p \in A_\alpha (p \Vdash \dot{f}(\check{\alpha}) = \check{\beta})\}.$$

Then each X_α is a subset of λ of size $< \lambda$. Let $X = \bigcup_{\alpha<\delta} X_\alpha$, which has size $< \lambda$. Since λ is regular in V, there is $\beta < \lambda$ be such that $X \subseteq \beta$. We claim $1 \Vdash \operatorname{ran} \dot{f} \subseteq \check{X}$. For if not, then there is $p \in \mathbb{P}$ and $\alpha < \delta$ such that $p \Vdash \dot{f}(\check{\alpha}) \notin \check{X}$. But there is $a \in A_\alpha$ such that p is compatible with a, and $a \Vdash \dot{f}(\check{\alpha}) \in \check{X}$, a contradiction. $\square$

Lemma 2.10 (Easton)**.** *Suppose $\mathbb{P}$ is κ-c.c., $\mathbb{Q}$ is κ-distributive, and $\Vdash_{\mathbb{Q}} \check{\mathbb{P}}$ is κ-c.c. Then $\Vdash_{\mathbb{P}}$ is $\check{\mathbb{Q}}$ is κ-distributive.*

Proof. Let $G \times H$ be $(\mathbb{P} \times \mathbb{Q})$-generic over V. Let $\mu < \kappa$ and let $f : \mu \to \mathrm{Ord}$ be in $V[G][H]$. We must show that $f \in V[G]$. Let $\tau \in V[H]$ be a $\mathbb{P}$-name for f. For each $\alpha < \mu$, let $A_\alpha \subseteq \mathbb{P}$ be a maximal antichain such that each $p \in A_\alpha$ decides $\tau(\check{\alpha})$ to be some ordinal β^p_α. Since $\mathbb{P}$ is κ-c.c. in $V[H]$, each A_α has size $< \kappa$. Let

$$\tau' = \{(p, \langle \check{\alpha}, \check{\beta}^p_\alpha \rangle) : \alpha < \delta \text{ and } p \in A_\alpha\}.$$

Then $1 \Vdash^{V[H]}_{\mathbb{P}} \tau' = \tau$. Otherwise, there is some $q \in \mathbb{P}$ and some $\alpha < \mu$ such that $q \Vdash \tau(\check{\alpha}) \neq \tau'(\check{\alpha})$, but q is compatible with some $p \in A_\alpha$, and by construction, $p \Vdash \tau'(\check{\alpha}) = \check{\beta}^p_\alpha = \tau(\check{\alpha})$. So we have that $(\tau')^G = \tau^G = f$. Now since every set in V is coded by a set of ordinals, and since $\tau' \subseteq V$ and $|\tau'| < \kappa$, we have that $\tau' \in V$ by the distributivity of $\mathbb{Q}$. Thus $(\tau')^G = f \in V[G]$. $\square$

The next lemma shows a key situation in which the hypotheses of the previous lemma hold.

Lemma 2.11. *If $\mathbb{Q}$ is κ-closed and $\mathbb{P}$ is κ-c.c., then $\Vdash_{\mathbb{Q}} \mathbb{P}$ is κ-c.c.*

Proof. If not, then some $q_0 \in \mathbb{Q}$ forces that there is an antichain of size κ contained in $\mathbb{P}$. Let $\dot{A}$ be a $\mathbb{Q}$-name for such an antichain, let $\dot{f}$ be a name for an injection from κ to $\dot{A}$. Recursively choose a descending sequence $\langle q_\alpha : \alpha < \kappa \rangle$ starting with q_0 as above, such that for each α, $q_{\alpha+1} \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha$, for some $p_\alpha \in \mathbb{P}$. Then $\langle p_\alpha : \alpha < \kappa \rangle$ is an antichain in $\mathbb{P}$, since for $\alpha < \beta < \kappa$, $q_{\beta+1} \Vdash \check{p}_\beta \perp \check{p}_\alpha$. But the ordering on $\mathbb{P}$ is determined in V, so $p_\alpha \perp p_\beta$. This contradicts the κ-c.c. of $\mathbb{P}$. $\square$

Lemma 2.12. *Suppose $\mathbb{P}$ is a κ-c.c. partial order of size λ. Let μ, δ be cardinals. Then $\mathbb{P}$ forces that*

$$\left(\mu^\delta\right)^{V[G]} \leq \left((\lambda \cdot \mu)^{<\kappa})^\delta\right)^V.$$

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name for a function from δ to μ. For each $\alpha < \delta$, let A_α be a maximal antichain of conditions deciding $\dot{f}(\alpha)$. For each such A_α, there are $\mu^{<\kappa}$ possibilities for the function $g_\alpha : A_\alpha \to \mu$ defined by $g(p) =$ the value β such that $p \Vdash \dot{f}(\alpha) = \beta$. We construct a name τ by putting:

$$\tau = \bigcup_{\alpha < \delta} \{(p, \langle \alpha, g_\alpha(p) \rangle) : p \in A_\alpha\}.$$

This is forced to be equal to $\dot{f}$ by similar arguments as before. Now to build such a name directly, we would choose for each $\alpha < \delta$, an antichain A_α and a function $g_\alpha : A_\alpha \to \mu$. There are at most $\lambda^{<\kappa} \cdot \mu^{<\kappa} = (\lambda \cdot \mu)^{<\kappa}$ many choices for each α, and so at most $((\lambda \cdot \mu)^{<\kappa})^\delta$ many choices for the whole sequence. $\square$

2.2 Adding Cohen sets

Let κ be a regular cardinal and let X be a set. We define $\mathrm{Add}(\kappa, X)$ as the set of partial functions from $\kappa \times X$ to 2 of size $< \kappa$, partially ordered by $p \leq q$ when $p \supseteq q$.

Lemma 2.13. *Let κ be a regular cardinal and let X be a set.*

1. $\mathrm{Add}(\kappa, X)$ *is κ-closed.*
2. $\mathrm{Add}(\kappa, X)$ *is $(2^{<\kappa})^+$-c.c.*
3. $\mathrm{Add}(\kappa, X)$ *forces that $2^\kappa \geq |X|$.*

Proof. (1) is easy. For (2), let $\langle p_\alpha : \alpha < (2^{<\kappa})^+\rangle \subseteq \mathrm{Add}(\kappa, X)$.

Case 1: $D = \bigcup_\alpha \operatorname{dom} p_\alpha$ is a set of size $\leq 2^{<\kappa}$. Since κ is regular, $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. So there is a set $Y \subseteq (2^{<\kappa})^+$ of size $(2^{<\kappa})^+$ and a set $d \in D^{<\kappa}$ such that for all $\alpha \in X$, $\operatorname{dom} p_\alpha = d$. There are at most $2^{<\kappa}$ possibilities for $p_\alpha \restriction d$, so there is $Y' \subseteq Y$ of size $(2^{<\kappa})^+$ and a single $q \in \mathrm{Add}(\kappa, X)$ such that $p_\alpha = q$ for all $\alpha \in Y'$. So in particular, the p_α's do not form an antichain.

Case 2: D has size $(2^{<\kappa})^+$. Enumerate D as $\langle x_i : i < (2^{<\kappa})^+\rangle$. There is an ordinal $\eta < \kappa$ and a set $Y \subseteq (2^{<\kappa})^+$ of full size such that for all $\alpha \in Y$, the ordertype of $\{i : x_i \in \operatorname{dom} p_\alpha\}$ is η. Let $d_\alpha = \operatorname{dom} p_\alpha$, and let $\langle d_\alpha(i) : i < \eta\rangle$ enumerate d_α in increasing order.

Case 2a: For all $i < \eta$, $\sup\{j < (2^{<\kappa})^+ : d_\alpha(i) = x_j\} < (2^{<\kappa})^+\}$. Then by the regularity of $(2^{<\kappa})^+$, $\bigcup_{\alpha\in Y} d_\alpha$ has size $\leq 2^{<\kappa}$. This puts us back in Case 1.

Case 2b: There is $i < \eta$ such that $\sup\{j < (2^{<\kappa})^+ : d_\alpha(i) = x_j\} = (2^{<\kappa})^+$. Let ξ be the least such i. Then $\{d_\alpha(i) : i < \xi \text{ and } \alpha \in Y\}$ has size $\leq 2^{<\kappa}$. Thus there is a set $Y' \subseteq Y$ of full size and a set d such that $d_\alpha \restriction \xi = d$ for all $\alpha \in Y'$. Let $Y'' \subseteq Y'$ and $q \in \mathrm{Add}(\kappa, X)$ be such that Y'' is of full size and $p_\alpha \restriction d = q$ for all $\alpha \in Y''$. Let $\alpha \in Y''$ be arbitrary. Let $i_\alpha = \sup\{j : \exists i < \eta(d_\alpha(i) = x_j\}$. Let $\alpha' > \alpha$ be such that $d_\alpha(\xi)$ has index greater than i_α. Then $p_\alpha \cup p_{\alpha'}$ is a common extension of $p_\alpha, p_{\alpha'}$, so the collection was not an antichain. (Note that we can continue this recursively to find a set of full size of pairwise-compatible conditions.)

For (3), first note that if $G \subseteq \mathrm{Add}(\kappa, X)$ is a generic filter, then $\bigcup G$ is a function from $\kappa \times X$ to 2. We claim that for $x \neq y$ in X, $G \restriction \kappa \times \{x\} \neq G \restriction \kappa \times \{y\}$. For let $p \in \mathrm{Add}(\kappa, X)$ be arbitrary. Since $|\operatorname{dom} p| < \kappa$, let $\gamma < \kappa$ be such that p is undefined at both (γ, x) and (γ, y). Let $p' \leq p$ be such that $p'(\gamma, x) \neq p'(\gamma, y)$. Thus densely often, conditions force the slices of G at x and y to be distinct functions. $\square$

Exercise 2.14. *Use Lemma 2.12 to compute the exact value of 2^κ that is forced by* $\mathrm{Add}(\kappa, X)$.

Thus under GCH, $\mathrm{Add}(\kappa, X)$ is κ-closed and κ^+-c.c., and thus preserves all regular cardinals. This is not so if GCH fails just below κ:

Proposition 2.15. *For all regular cardinals κ and all nonempty X,* $\mathrm{Add}(\kappa, X)$ *forces $2^{<\kappa} = \kappa$. Thus if $2^\mu > \kappa$ for some $\mu < \kappa$, then* $\mathrm{Add}(\kappa, X)$ *collapses the cardinal 2^μ.*

Proof. Let $x \in X$. If $G \subseteq \mathrm{Add}(\kappa, X)$ is generic and $\alpha, \beta < \kappa$ let $f_{\alpha,\beta} : \beta \to 2$ be the function defined by $f_{\alpha,\beta}(i) = G(\alpha + i, x)$. In the generic extension, there is a bijection between these functions and κ. Let $p \in \mathrm{Add}(\kappa, X)$ and

$f \in 2^{<\kappa}$ be arbitrary. Let $\xi < \kappa$ be such that $p(i, x)$ is undefined for $i \geq \xi$. Let $p' \leq p$ be such that $p'(\xi + i, x) = f(i)$ for $i \in \operatorname{dom} f$. Then p' forces that $f \in \{f_{\alpha,\beta} : \alpha, \beta < \kappa\}$. Thus this collection is forced to include all of $2^{<\kappa}$. $\square$

2.3 Proof of Easton's Theorem

We say a set of ordinals X is an *Easton* set if for all regular cardinals κ, $|X \cap \kappa| < \kappa$. For a set of ordinals Z, we define the *Easton ideal* on Z as the set of all Easton subsets of Z. We say an ideal is κ-complete for a cardinal κ if it is closed under unions of size $< \kappa$.

Lemma 2.16. *Suppose Z is a set of ordinals, and κ is a regular cardinal such that $\kappa \leq \min(Z)$. Then the Easton ideal on Z is κ^+-complete.*

Proof. Fix a regular cardinals $\mu > \kappa$. Let $\langle X_i : i < \delta < \mu \rangle$ be Easton subsets of Z. For each i, $X_i \cap \mu$ is a set of size $< \mu$, so the set $\mu \cap \bigcup_i X_i = \bigcup_i (X_i \cap \mu)$ has size $< \mu$. $\square$

Let F be a function satisfying the hypotheses of Easton's Theorem. Let $Z \subseteq \operatorname{dom} F$ and let E be the Easton ideal on Z. Consider the partial order:

$$\mathbb{P}_Z := \prod_{\kappa \in Z}^{E} \operatorname{Add}(\kappa, F(\kappa)).$$

Let $\mathbb{P}$ denote $\mathbb{P}_{\operatorname{dom}(F)}$. For an ordinal α, let $\mathbb{P}_{<\alpha}$ denote $\mathbb{P}_{\operatorname{dom}(F) \cap \alpha}$, and let $\mathbb{P}_{\leq \alpha}$, $\mathbb{P}_{>\alpha}$, $\mathbb{P}_{\geq \alpha}$ denote the obvious analogous things.

Lemma 2.17. *Suppose Z, F are as above and $\kappa \leq \min(Z)$. Then $\mathbb{P}_Z$ is κ-closed.*

Proof. Suppose $\langle p_i : i < \delta < \kappa \rangle \subseteq \mathbb{P}_Z$. For each $\alpha \in Z$, there is a greatest lower bound $p_\delta(\alpha)$ for the sequence $\langle p_i(\alpha) : i < \delta < \kappa \rangle$. Since the support of each p_i is Easton and $E \restriction Z$ is κ-complete, the sequence $\langle p_\delta(\alpha) : \alpha \in Z \rangle$ is a member of $\prod_{\alpha \in Z}^{E} \operatorname{Add}(\alpha, F(\alpha))$ and a lower bound to the sequence $\langle p_i : i < \delta \rangle$. $\square$

Lemma 2.18. *If κ is a regular cardinal, then $\mathbb{P}_{\leq \kappa}$ has the κ^+-c.c.*

Proof. Any function from ordinals to functions on n-tuples of ordinals can be coded as a function on $(n+1)$-tuples of ordinals. Thus $\mathbb{P}_{\leq \kappa}$ can be regarded as a collection of partial functions of size $< \kappa$ on θ^3 to 2 for some ordinal θ. More specifically, if $\theta = \max\{\kappa, \sup_{\alpha \leq \kappa} F(\alpha)\}$, it is the set of partial functions p on θ^3 to 2 such that:

1. $\{\alpha : \exists \beta \exists \gamma (\alpha, \beta, \gamma \in \operatorname{dom} p)\}$ is an Easton subset of $(\kappa + 1) \cap \operatorname{dom} F$.
2. For all α, $\{\beta : \exists \gamma (\alpha, \beta, \gamma \in \operatorname{dom} p\}$ is a bounded subset of α.
3. For all α and β, $\{\gamma : (\alpha, \beta, \gamma \in \operatorname{dom} p\}$ is a subset of $F(\alpha)$ of size $< \alpha$.

The ordering is $p \le q$ when $p \supseteq q$. The family has the property that if p, q agree on $\operatorname{dom} p \cap \operatorname{dom} q$, then $p \cup q$ is in the family. Thus there is an order and antichain preserving embedding of $\mathbb{P}_{\le\kappa}$ into $\operatorname{Add}(\kappa, \theta)$. The latter has the $(2^{<\kappa})^+ = \kappa^+$-c.c., so $\mathbb{P}_{\le\kappa}$ does also. □

Lemma 2.19. *$\mathbb{P}$ preserves regular cardinals.*

Proof. Suppose κ is a regular cardinal in V. Suppose to the contrary that there is some condition p and an ordinal $\delta < \kappa$ such that $p \Vdash \operatorname{cf}(\check{\kappa}) = \check{\delta}$. Since δ is a cofinality, it must be regular in V. Write $\mathbb{P}$ as $\mathbb{P}_{\le\delta} \times \mathbb{P}_{>\delta}$, and let p_0, p_1 be the projections of p to the respective factors. Let G_1 be $\mathbb{P}_{>\delta}$-generic over V with $p_1 \in G_1$. Since $\mathbb{P}_{>\delta}$ is δ^+-closed, $V[G_1] \models \operatorname{cf}(\kappa) > \delta$. Since $\mathbb{P}_{\le\delta}$ is δ^+-c.c. in $V[G_1]$, forcing with it over $V[G_1]$ preserves that $\operatorname{cf}(\kappa) > \delta$. Taking $G_0 \subseteq \mathbb{P}_{\le\delta}$ generic over $V[G_1]$ with $p_0 \in G_0$, we get a contradiction to what we assumed p forces. □

Lemma 2.20. *For each $\kappa \in \operatorname{dom} F$, $\mathbb{P}$ forces that $2^\kappa = F(\kappa)$.*

Proof. If $\kappa \in \operatorname{dom} F$, then since $\mathbb{P}$ projects onto $\operatorname{Add}(\kappa, F(\kappa))$, $\mathbb{P}$ forces that there is an injection from $F(\kappa)$ to 2^κ. $\mathbb{P}_{\le\kappa}$ is κ^+-c.c. and $\mathbb{P}_{>\kappa}$ is κ^+-closed. By Easton's Lemma, $\mathbb{P}_{>\kappa}$ adds no subsets of κ, even after forcing with $\mathbb{P}_{\le\kappa}$. So it suffices to show that the inequality $2^\kappa \le F(\kappa)$ is forced by $\mathbb{P}_{\le\kappa}$.

Note that $|\mathbb{P}_{\le\kappa}| = F(\kappa)^{<\kappa} = F(\kappa)$, by GCH and the fact that $\operatorname{cf}(F(\kappa)) > \kappa$. Let $G \subseteq \mathbb{P}_{\le\kappa}$ be generic. By Lemma 2.12,

$$(2^\kappa)^{V[G]} \le \left(((F(\kappa) \cdot 2)^{<\kappa^+})^\kappa \right)^V = (F(\kappa)^\kappa)^V = F(\kappa).$$

□

This concludes the proof of Easton's Theorem.

What happens to the powers of cardinals not in $\operatorname{dom} F$? Suppose κ is such a cardinal. If κ is regular, then we can extend F to F' by setting $F'(\kappa)$ to be the least possible value that yields a function satisfying the hypotheses of Easton's Theorem. Let $\theta = \max\{\kappa^+, \sup_{\alpha<\kappa} F(\alpha)\}$. If $\operatorname{cf}(\theta) \le \kappa$, then $F'(\kappa) = \theta^+$. Otherwise, $\operatorname{cf}(\theta) > \kappa$ and $F'(\kappa) = \theta$. If $\mathbb{P}'$ is the Easton forcing defined via F', then if G' is $\mathbb{P}'$-generic over V, $V[G']$ has the same cardinals as V and satisfies $2^\kappa = F'(\kappa)$. $\mathbb{P}'$ canonically projects to $\mathbb{P}$, yielding $\mathbb{P}$-generic G. In $V[G]$, we must have $2^\kappa \le F'(\kappa)$. By the monotonicity of the power function and König's Theorem, this is the smallest possible value, so $V[G] \models 2^\kappa = F(\kappa)$.

What if κ is singular? If $\operatorname{dom} F$ is bounded below κ, then we may compute as before. For some $\mu < \kappa$, $\mathbb{P}_{<\kappa}$ has the μ^+-c.c., so if G is $\mathbb{P}_{<\kappa}$-generic, then

$$(2^\kappa)^{V[G]} \le (|\mathbb{P}_{<\kappa}|^\mu)^\kappa = |\mathbb{P}_{<\kappa}|^\kappa \le (\sup_{\alpha<\kappa} F(\alpha))^\kappa \le ((2^{<\kappa})^\kappa)^{V[G]} = (2^\kappa)^{V[G]}.$$

In the above inequalities, we mean to compute the unrelativized terms in V. By Easton's Lemma, $\mathbb{P}_{>\kappa}$ adds no subsets of κ after forcing with $\mathbb{P}_{<\kappa}$.

If $\operatorname{dom} F$ is unbounded below κ, then $\mathbb{P}_{<\kappa}$ does not have the κ^+-c.c., so the name-counting argument alone doesn't give enough information. We need the following:

Lemma 2.21. *For all infinite cardinals κ, $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.*

Proof. Let $\langle \kappa_i : i < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence with limit κ.

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa$$

□

We split $\mathbb{P}$ as $\mathbb{P}_{\leq \mathrm{cf}(\kappa)} \times \mathbb{P}_{>\mathrm{cf}(\kappa)}$. Let $\theta = \sup_{\alpha<\kappa} F(\alpha)$. We know that $\mathbb{P}$ forces $2^{<\kappa} = \theta$. So it suffices to compute the value forced for $\theta^{\mathrm{cf}(\kappa)}$. By Easton's Lemma, it suffices to compute the value forced by $\mathbb{P}_{\leq \mathrm{cf}(\kappa)}$. If G is $\mathbb{P}_{\leq \mathrm{cf}(\kappa)}$-generic, then

$$(\theta^{\mathrm{cf}(\kappa)})^{V[G]} \leq ((\theta \cdot |\mathbb{P}_{\leq \mathrm{cf}(\kappa)}|)^{\mathrm{cf}(\kappa)})^{\mathrm{cf}(\kappa)} = \theta^{\mathrm{cf}(\kappa)}.$$

The latter two terms are computed in V. There are two cases. In the first case, $F(\alpha)$ is not eventually constant below κ, so $\mathrm{cf}(\theta) = \mathrm{cf}(\kappa)$. Then $\theta^{\mathrm{cf}(\kappa)} = \theta^+$. In the second case, θ is the eventual value of $F(\alpha)$, and by the requirements on F, $\mathrm{cf}(\theta) > \kappa$. So in this case, $\theta^{\mathrm{cf}(\kappa)} = \theta$.

The point is that if F is such that κ is forced to be a singular strong limit, then it is forced that $2^\kappa = \kappa^+$. Producing a model where this fails requires large cardinals beyond measurable.

3 Large cardinals

3.1 Normal ideals

Suppose $Z \subseteq \mathcal{P}(X)$. For collections of subsets of Z indexed by X, $\langle A_x : x \in X\rangle$, we define the following operations. The *diagonal intersection:*

$$\Delta_{x\in X} A_x := \{z \in Z : \forall x \in z (z \in A_x)\}$$

The *diagonal union:*

$$\nabla_{x\in X} A_x := \{z \in Z : \exists x \in z (z \in A_x)\}$$

An ideal I on Z is called *normal* when it is closed under diagonal unions, or equivalently, when its dual filter I^* is closed under diagonal intersections. We say a set $A \subseteq Z$ is I-measure-zero when $A \in I$, I-measure-one when $A \in I^*$, and I-positive when $A \notin I$. I-positive is equivalent to having nonempty intersection with each I-measure-one set.

Suppose $Z \subseteq \mathcal{P}(X)$, $n < \omega$, and $|X| \geq n$. Let $\langle A_{\vec{x}} : \vec{x} \in X^n\rangle \subseteq \mathcal{P}(Z)$. We define the diagonal intersection and union of sets indexed by n-tuples:

$$\Delta_{\vec{x}\in X^n} A_{\vec{x}} := \{z : \forall \vec{x} \in z^n (z \in A_{\vec{x}})\}$$

$$\nabla_{\vec{x}\in X^n} A_{\vec{x}} := \{z : \exists \vec{x} \in z^n (z \in A_{\vec{x}})\}$$

Lemma 3.1 (Fodor)**.** *Suppose X is an infinite set, $Z \subseteq \mathcal{P}(X)$, and I is an ideal on Z. The following are equivalent:*

1. *I is normal.*
2. *For each $n < \omega$, I is closed under diagonal unions indexed by n-tuples.*
3. *For each $n < \omega$, each I-positive $A \subseteq Z$, and each function $f : A \to X^n$ such that $f(z) \in z^n$ for all $z \in A$, there is an I-positive $B \subseteq A$ on which f is constant.*

Proof. Suppose I is normal. We prove (2) by induction on n. For $n = 1$, it is true by definition. Suppose it is true for n. Let $\langle A_{\vec{x}} : x \in X^{n+1} \rangle \subseteq I$. For each $x \in X$, let $B_x = \nabla_{\vec{y} \in X^n} A_{\langle x \rangle ^\frown \vec{y}}$. By the induction hypothesis, each B_x is in I. Let $C = \nabla_{x \in X} B_x$. Then $C \in I$. Note that

$$\begin{aligned} C &= \{z : \exists x \in z(z \in B_x)\} = \{z : \exists x \in z(\exists \vec{y} \in z^n(z \in A_{x^\frown \vec{y}}))\} \\ &= \{z : \exists \vec{v} \in z^{n+1}(z \in A_{\vec{v}})\} = \nabla_{\vec{v} \in X^{n+1}} A_{\vec{v}}. \end{aligned}$$

Now suppose I is closed under diagonal intersection by n-tuples. Suppose A is I-positive and $f : A \to X^n$ is such that $f(z) \in z^n$ for each $z \in A$. Assume towards a contradiction that there is no I-positive set on which f is constant. Then for each $\vec{x} \in X^n$, $f^{-1}[\{\vec{x}\}] \in I$, and so $\nabla_{\vec{x} \in X^n} f^{-1}[\{\vec{x}\}] \in I$. But

$$\nabla_{\vec{x} \in X^n} f^{-1}[\{\vec{x}\}] = \{z : \exists \vec{x} \in z^n(f(z) = \vec{x})\} = A,$$

which is a contradiction.

Suppose that (3) holds for $n = 1$. Suppose $\langle A_x : x \in X \rangle \subseteq I$. If $\nabla A_x \notin I$, then it is I-positive. Let $f : \nabla A_x \to X$ be such that for each $z \in \nabla A_x$, $f(z) \in z \in A_{f(z)}$. Let $B \subseteq \nabla A_x$ be I-positive on which f is constant with value x_0. Then $B \subseteq A_{x_0}$, which is impossible since $B \notin I$. □

The functions in clause (3) of Fodor's Lemma are called *regressive*.

An important example of a normal ideal is the *nonstationary ideal*, whose dual filter is called the *club filter*. A set $C \subseteq Z \subseteq \mathcal{P}(X)$ is called a *closed unbounded* subset of Z, or a "club" subset of Z, when there is some function $f : X^{<\omega} \to X$ such that $C = \{z : f[z^{<\omega}] \subseteq z\}$, the set of z that are closed under f. We sometimes denote this set by C_f. The club filter on Z is the collection of all supersets of clubs. A set is called *stationary* when it has nonempty intersection with every club. The dual ideal to the club filter is the collection of all nonstationary sets.

An ideal I on $Z \subseteq \mathcal{P}(X)$ is called *fine* if for all $x \in X$, $\hat{x} := \{z : x \in z\} \in I^*$.

Lemma 3.2. *For any $Z \subseteq \mathcal{P}(X)$, the nonstationary ideal on Z is normal and fine.*

Proof. For fineness, let $x_0 \in X$, and let $f : Z \to X$ be constant with value x_0. Then $C_f = \hat{x}_0$.

For normality, let $\langle A_x : x \in X \rangle$ be a sequence of sets in the club filter, and for each $x \in X$, let $f_x : X^{<\omega} \to X$ be such that $A \supseteq C_{f_x}$. Fix some $x_0 \in X$. Define $f : X^{<\omega} \to X$ by $f(\vec{v}) = x_0$ for $\vec{v}$ of length 1, and $f(\langle v_0, v_1, \dots, v_n \rangle) = f_{v_0}(\langle v_1, \dots, v_n \rangle)$ for $\vec{v}$ of longer length. If z is closed under f, then for all $x \in z$, z is closed under f_x. Thus $C_f \subseteq \nabla A_x$. $\square$

Lemma 3.3. *Normal fine ideals are countably complete.*

Proof. Let I be a normal fine ideal on $Z \subseteq \mathcal{P}(X)$. First consider the case that X is finite. Then Z and $\mathcal{P}(Z)$ are finite, so we can take the union of all members of I to find a maximal element. Thus I is κ-complete for all κ.

Suppose X is infinite, and let $\langle x_i : i < \omega \rangle$ enumerate distinct elements of X. First we claim that the set of $z \in Z$ such that $\{x_i : i < \omega\} \subseteq z$ is I-measure-one. Suppose towards a contradiction that this fails, so that the set $B = \{z : \{x_i : i < \omega\} \nsubseteq z\}$ is I-positive. Let $C = \hat{x}_0 \cap B$, which is also I-positive. For each $z \in B$, there is a largest n such that $\{x_0, \dots, x_n\} \subseteq z$. Call this n_z. Let $f : B \to X$ be such that $f(z) = x_{n_z}$. f is regressive on an I-positive set, so there is an I-positive $C \subseteq B$ and an $m < \omega$ such that $f(z) = x_m$ for all $z \in C$. This implies that for all $z \in C$, $x_{m+1} \notin z$. But $\hat{x}_{m+1} \cap C$ is I-positive, a contradiction.

Now let $\langle A_i : i < \omega \rangle \subseteq I$. Suppose towards a contradiction that $B = \bigcup_{i<\omega} A_i \notin I$. Let $C = \{z \in B : \{x_i : i < \omega\} \subseteq z\}$, which is also I-positive. For $z \in C$, let $f(z) = x_n$, where n is such that $z \in A_n$. Then f has a constant value x_m on an I-positive set $D \subseteq C$. Then $D \subseteq A_m$, a contradiction. $\square$

Question. *Is it possible to have an ideal which is normal but not countably complete?*

Corollary 3.4. *Suppose I is a normal fine ideal on $Z \subseteq \mathcal{P}(X)$. Then I is closed under diagonal unions indexed by all finite subsets of X. Furthermore, if A is I-positive and $f : A \to X^{<\omega}$ is such that $f(z) \in z^{<\omega}$ for all $z \in A$, then there is an I-positive $B \subseteq A$ such that f is constant on B.*

Proof. Suppose $\langle A_{\vec{x}} : \vec{x} \in X^{<\omega} \rangle \subseteq I$. For each $n < \omega$, let $B_n = \nabla_{\vec{x} \in X^n} A_{\vec{x}}$. Then each B_n is in I. By countable completness, $\bigcup_{n<\omega} B_n \in I$, and $\bigcup B_n = \{z : \exists \vec{x} \in z^{<\omega} (z \in A_{\vec{x}})\} = \nabla_{\vec{x} \in X^{<\omega}} A_{\vec{x}}$.

Now suppose A is I-positive and $f : A \to X^{<\omega}$ is regressive. By countable completeness, there is some $n < \omega$ and some I-positive $B \subseteq A$ such that $f(z) \in z^n$ for all $z \in B$. By Fodor's Lemma, there is some I-positive $C \subseteq B$ such that f is constant on C. $\square$

Theorem 3.5. *For any $Z \subseteq \mathcal{P}(X)$, the nonstationary ideal on Z is the smallest normal fine ideal.*

Proof. Let I be a normal fine ideal on $Z \subseteq \mathcal{P}(X)$. Suppose towards a contradiction that there is some nonstationary set that is not in I. Then there is some $f : X^{<\omega} \to X$ and an I-positive $A \subseteq Z$ such that all $z \in A$ are *not* closed under f. For $z \in A$, let $g(z) \in z^{<\omega}$ be such that $f(g(z)) \notin z$. By the above corollary, there is an I-positive $C \subseteq B$ and a $\vec{v} \in X^{<\omega}$ such that $g(z) = \vec{v}$ for all $z \in C$. Thus for all $z \in C$, $f(\vec{v}) \notin z$. This contradicts fineness. □

If I is an ideal on a set Z, then $\mathcal{P}(Z)/I$ is a Boolean algebra under the ordinary set operations, modulo I. If I is normal, then the algebra enjoys some degree of completeness:

Lemma 3.6. *Suppose I is a normal fine ideal on $Z \subseteq \mathcal{P}(X)$. If $\langle A_x : x \in X \rangle$ is a sequence of subsets of Z, then $[\nabla A_x]_I$ is the least upper bound to the set $\{[A_x]_I : x \in X\}$ in the Boolean algebra $\mathcal{P}(Z)/I$.*

Proof. Note that for any $x \in X$, $A_x =_I A_x \cap \hat{x}$. $z \in A_x \cap \hat{x}$ iff $x \in z \in A_x$, which implies $z \in \nabla_y A_y$. Thus $A_x =_I A_x \cap \hat{x} \subseteq \nabla_y A_y$, so $[\nabla_y A_y]_I$ is an upper bound to the collection in $\mathcal{P}(Z)/I$.

To show it is the least upper bound, suppose $B \subseteq \nabla A_x$ is I-positive. Then for each $z \in B$, there is $x \in z$ such that $z \in A_x$. Let $f : B \to X$ be a regressive function that chooses such witnesses. Let $C \subseteq B$ be I-positive on which f is constant with value x_0. Then C is an I-positive subset of A_{x_0}. Therefore, every element less than $[\nabla_x A_x]_I$ is compatible with some $[A_y]_I$. This means there cannot be a strictly smaller upper bound $[D]_I$ to the collection, since in that case, $[\nabla A_x \setminus D]_I$ would be a positive element less than $[\nabla A_x]_I$ and incompatible with all $[A_y]_I$. □

Corollary 3.7. *If I is a normal fine ideal on $Z \subseteq \mathcal{P}(X)$, and $\mathcal{P}(Z)/I$ has the $|X|^+$-c.c., then $\mathcal{P}(Z)/I$ is a complete Boolean algebra.*

3.2 Measurable cardinals

Definition. *A cardinal $\kappa > \omega$ is measurable when there is an ultrafilter $\mathcal{U}$ over κ such that $\mathcal{U}$ is nonprincipal (no singleton is in $\mathcal{U}$) and κ-complete ($\mathcal{U}$ is closed under $<\kappa$-sized intersections).*

We will assume all ultrafilters considered below to be nonprincipal.

Exercise 3.8. *Show that there exists a measurable cardinal iff there exists an ω_1-complete ultrafilter over some set.*

Lemma 3.9. *Measurable cardinals are regular.*

Proof. Let $\mathcal{U}$ be a κ-complete nonprincipal ultrafilter on κ. Suppose $\delta < \kappa$ and $\langle \alpha_i : i < \delta \rangle \subseteq \kappa$. Since $\mathcal{U}$ is κ-complete and nonprincipal, for each $i < \delta$, $A_i := \{\beta < \kappa : \beta > \alpha_i\} \in \mathcal{U}$. $\bigcup_{i<\delta} A_i \in \mathcal{U}$, and thus there is some $\alpha < \kappa$ such that $\alpha > \alpha_i$ for all $i < \delta$. □

Lemma 3.10. *Measurable cardinals are strongly inaccessible.*

Proof. Suppose otherwise. Let $\delta < \kappa$ and let $\langle f_\alpha : \alpha < \kappa \rangle$ be pairwise distinct functions from δ to 2. For each $\alpha < \delta$, there is $i_\alpha < 2$ such that

$$A_\alpha := \{\beta < \kappa : f_\beta(\alpha) = i_\alpha\} \in \mathcal{U}.$$

Let $A = \bigcap_{\alpha<\delta} A_\alpha$. Let $\beta_0 < \beta_1$ be in A. Then for each $\alpha < \delta$, $f_{\beta_0}(\alpha) = f_{\beta_1}(\alpha) = i_\alpha$. Thus $f_{\beta_0} = f_{\beta_1}$, a contradiction. □

Suppose κ is measurable with witnessing measure $\mathcal{U}$. Then we can form the ultrapower of the universe, $V^\kappa/\mathcal{U}$. Since $\mathcal{U}$ is countably complete, there cannot be a descending chain in the membership relation of $V^\kappa/\mathcal{U}$. For suppose otherwise, and let $\langle f_n : n < \omega\rangle$ be functions on κ such that for each n, $A_n := \{\alpha : f_{n+1}(\alpha) \in f_n(\alpha)\} \in \mathcal{U}$. Then there is $\alpha \in \bigcap_{n<\omega} A_n$, and we have that $f_0(\alpha) \ni f_1(\alpha) \ni \cdots \ni f_n(\alpha) \ni \dots$. This contradicts that membership is well-founded in V.

By Mostowski's collapsing lemma, $V^\kappa/\mathcal{U}$ is isomorphic to a transitive class M. Let $\pi : V^\kappa/\mathcal{U} \to M$ be the transitive collapse map. For each $x \in V$, let c_x be the constant function on κ with value x. Then there is an elementary embedding $j : V \to M$ given by:

$$j(x) = \pi([c_x]_\mathcal{U}).$$

Lemma 3.11. *Suppose $j : V \to M$ is derived from a κ-complete ultrafilter on κ. Then the least ordinal moved by j, called the* critical point*, is κ.*

Proof. First note that for each $\alpha < \kappa$, the identity function on κ dominates c_α modulo $\mathcal{U}$. Thus the ordertype of the set of ordinals below $j(\kappa)$ is at least $\kappa + 1$, so $j(\kappa) > \kappa$. Now suppose inductively that for some $\beta < \kappa$, $j(\alpha) = \alpha$ for each $\alpha < \beta$. Suppose $[f]_\mathcal{U} < [c_\beta]_\mathcal{U}$. Then $f(\gamma) < \beta$ for all γ in a set $X \in \mathcal{U}$. By κ-completeness, there is some α_0 such that $\{\gamma \in X : f(\gamma) = \alpha_0\} \in \mathcal{U}$, so $[f]_\mathcal{U} = [c_{\alpha_0}]_\mathcal{U}$. Thus the ordertype of the set of ordinals below $j(\beta)$ is β, so $j(\beta) = \beta$. □

Theorem 3.12. *The following are equivalent:*

1. *κ is measurable.*
2. *There is an elementary embedding $j : V \to M$, definable from parameters with M a transitive class, having critical point κ.*
3. *There is a transitive set N and an elementary embedding $j : V_{\kappa+1} \to N$ with critical point κ.*

Proof. (1) ⇒ (2) is via the ultrapower construction, and clearly (2) ⇒ (3). To show (3) ⇒ (1), assume we have such a map j. Let $\mathcal{U} = \{X \subseteq \kappa : \kappa \in j(X)\}$. By the elementarity of j, $\mathcal{U}$ is a filter. If $\alpha < \kappa$, then $\kappa \notin j(\{\alpha\}) = \{\alpha\}$, so it is nonprincipal. To show κ-completeness, suppose $\delta < \kappa$ and $\langle X_\alpha : \alpha < \delta\rangle \subseteq \mathcal{U}$. This sequence can be coded into a single subset of κ, so that it makes sense to talk about it in $V_{\kappa+1}$. $j(\langle X_\alpha : \alpha < \delta\rangle) = \langle j(X_\alpha) : \alpha < \delta\rangle$, and $M \models \kappa \in \bigcap_{\alpha<\delta} j(X_\alpha)$. By elementarity, $j(\bigcap_{\alpha<\delta} X_\alpha) = \bigcap_{\alpha<\delta} j(X_\alpha)$, so $\bigcap_{\alpha<\delta} X_\alpha \in \mathcal{U}$. □

Lemma 3.13. *Suppose $j : V \to M$ is derived from a κ-complete ultrafilter on κ. Let $\pi : V^\kappa/\mathcal{U}$ be the transitive collapse map. Then for all functions f with domain κ, $\pi([f]_\mathcal{U}) = j(f)(\pi([\mathrm{id}]_\mathcal{U})$.*

Proof. Let c_f be the constant function with value f on κ. Then for all $\alpha < \kappa$, $c_f(\alpha)(\text{id}(\alpha)) = f(\alpha)$. Thus $V^\kappa/\mathcal{U} \models [c_f]([\text{id}]) = [f]$. $\square$

Theorem 3.14. *Suppose $j : V \to M$ is derived from a κ-complete ultrafilter on κ. Then $M^\kappa \subseteq M$.*

Proof. Let $\langle x_\alpha : \alpha < \kappa \rangle \subseteq M$. Let $\pi : V^\kappa/\mathcal{U}$ be the transitive collapse map. For each α, let f_α be a function such that $\pi([f_\alpha]) = x_\alpha$. Let $\langle f'_\alpha : \alpha < j(\kappa)\rangle = j(\langle f_\alpha : \alpha < \kappa\rangle)$. In M, we can compute $\langle j(f_\alpha) : \alpha < \kappa\rangle$ by simply taking the restricted sequence $\langle f'_\alpha : \alpha < \kappa\rangle$. Finally note that:

$$\langle j(f_\alpha)(\pi([\text{id}])) : \alpha < \kappa\rangle = \langle \pi([f_\alpha]) : \alpha < \kappa\rangle = \langle x_\alpha : \alpha < \kappa\rangle.$$

The leftmost operation can be carried out in M since M has the objects $\pi([\text{id}])$ and $\langle j(f_\alpha) : \alpha < \kappa\rangle$. $\square$

Proposition 3.15. *Suppose $j : V \to M$ is derived from a κ-complete ultrafilter on κ. Then $M^{\kappa^+} \nsubseteq M$.*

Proof. We show that $j[\kappa^+] \notin M$. Suppose otherwise. Then $j[\kappa^+]$ is represented by a function f on κ. We may assume that for all $\alpha < \kappa$, $f(\alpha)$ is a subset of κ^+. Let $S = \{\alpha : |f(\alpha)| \leq \kappa\}$ and $B = \{\alpha : |f(\alpha)| > \kappa\}$.

Suppose first that $S \in \mathcal{U}$. Let $\gamma < \kappa^+$ be such that $\gamma \notin \bigcup_{\alpha\in S} f(\alpha)$. Then $[c_\gamma]_\mathcal{U} \notin [f]_\mathcal{U}$. But this contradicts that $j(\gamma) \in j[\kappa^+] = [f]_\mathcal{U}$. So we must have $B \in \mathcal{U}$. However, we inductively build a one-to-one function $g : \kappa \to \bigcup_{\alpha\in B} f(\alpha)$ such that for all $\alpha \in B$, $g(\alpha) \in f(\alpha)$. We just use the fact that each $f(\alpha)$ is large and we need to choose a small number of points. We have that $[g]_\mathcal{U} \in [f]_\mathcal{U}$. But $[g]_\mathcal{U}$ is not equal to any $j(\alpha)$ because g is not constant on a large set. $\square$

Lemma 3.16. *Suppose $\mathcal{U}$ is a normal κ-complete ultrafilter on κ. Then κ is represented in the ultrapower by the identity function.*

Proof. If $[f]_\mathcal{U} < [\text{id}]_\mathcal{U}$, then $f(\alpha) < \alpha$ on a set $X \in \mathcal{U}$. By normality, it is constant on some $Y \in \mathcal{U}$. Thus the ordertype of the set of ordinals below $[\text{id}]$ is κ. $\square$

Lemma 3.17. *Suppose $\mathcal{U}$ is derived from an elementary embedding $j : V \to M$. Then $\mathcal{U}$ is normal.*

Proof. By assumption, $\mathcal{U} = \{X \subseteq \kappa : \kappa \in j(X)\}$. If f is regressive on a set $X \in \mathcal{U}$, then there is some $\alpha < \kappa$ such that $j(f)(\kappa) = \alpha$. Thus $\{\beta : f(\beta) = \alpha\} \in \mathcal{U}$. $\square$

The following shows that many large cardinal properties weaker than measurability, such as strong inaccessibility, Mahloness, weak compactness, etc., reflect below a measurable.

Theorem 3.18. *Suppose κ is measurable, $\varphi(x, y)$ is a formula in the language of set theory, and $a \in V_\kappa$. If $V_{\kappa+1} \models \varphi(\kappa, a)$, then there is $\delta < \kappa$ with $a \in V_\delta$ such that $V_{\delta+1} \models \varphi(\delta, a)$.*

Proof. Let $\mathcal{U}$ be a normal ultrafilter on κ. Let $j : V \to M$ be derived from $\mathcal{U}$. Then $V^M_{\kappa+1} = V_{\kappa+1}$, so $M \models$ "$V_{\kappa+1} \models \varphi(\kappa, a)$." Since $\kappa = [\mathrm{id}]_{\mathcal{U}}$ and $j(a) = a$, there is a set $X \in \mathcal{U}$ such that for all $\delta \in X$, $V_{\delta+1} \models \varphi(\delta, a)$. □

Proposition 3.19. *Suppose κ is measurable and $2^\alpha = \alpha^+$ for all $\alpha < \kappa$. Then $2^\kappa = \kappa^+$.*

Proof. Let $j : V \to M$ be an elementary embedding derived from a normal ultrafilter on κ. By elementarity, $M \models$ "$\forall \alpha < j(\kappa)(2^\alpha = \alpha^+)$. Thus $M \models 2^\kappa = \kappa^+$. Because $M^\kappa \subseteq \kappa$, the bijection in M witnessing this statement is a bijection between $\mathcal{P}(\kappa)^V$ and $(\kappa^+)^V$. So $V \models 2^\kappa = \kappa^+$. □

Lemma 3.20. *If $j : V \to M$ is derived from a κ-complete ultrafilter on κ, then $2^\kappa < j(\kappa) < (2^\kappa)^+$.*

Proof. Since $\mathcal{P}(\kappa) \subseteq M$ and M thinks $j(\kappa)$ is inaccessible, $(2^\kappa)^V \leq (2^\kappa)^M < j(\kappa)$. Further, there are at most κ^κ ordinals below $j(\kappa)$, since every such ordinal is represented by the equivalence class of a function $f : \kappa \to \kappa$. Thus $j(\kappa) < (2^\kappa)^+$. □

Exercise 3.21. *Show that if $\mathcal{U}$ is a normal ultrafilter on κ, $j : V \to M$ is the embedding derived from $\mathcal{U}$, and $\mathcal{U}'$ is the ultrafilter derived from j, then $\mathcal{U}' = \mathcal{U}$.*

Exercise 3.22. *Show that if $\mathcal{U}$ is a normal ultrafilter on κ, $j : V \to M$ is the embedding derived from $\mathcal{U}$, then $\mathcal{U} \notin M$.*

3.3 Measurability and GCH: Kunen-Paris Theorem

Lemma 3.23 (Silver)**.** *Suppose $M \models$ ZFC and $j : M \to N$ is an elementary embedding. Suppose $\mathbb{P} \in M$ is a partial order. If there are filters $G \subseteq \mathbb{P}$ and $H \subseteq j(\mathbb{P})$ such that G is generic over M, H is generic over N, and $j[G] \subseteq H$, then j can be extended to an elementary embedding $\hat{j} : M[G] \to N[H]$.*

Proof. Suppose $M[G] \models \varphi(\tau_1^G, \dots, \tau_n^G)$, where $\tau_1, \dots, \tau_n$ are $\mathbb{P}$-names from M. Then there is $p \in G$ such that $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$. By the elementarity of j, $j(p) \Vdash^N_{j(\mathbb{P})} \varphi(j(\tau_1), \dots, j(\tau_n))$. Since $j(p) \in H$ and H is generic over N, $N[H] \models \varphi(j(\tau_1)^H, \dots, j(\tau_n)^H)$. Thus if we define $\hat{j}(\tau^G) = j(\tau)^H$ for all $\mathbb{P}$-names $\tau \in M$, then $\hat{j}$ is an elementary embedding from $M[G]$ to $N[H]$. □

Remark 3.24. *A kind of converse to the above theorem holds. If G is $\mathbb{P}$-generic over M, then j can be extended to $\hat{j} : M[G] \to N'$, where N is a transitive subclass of N', iff we can find $H \subseteq j(\mathbb{P})$ generic over N with $N' = N[H]$ and $j[G] \subseteq H$.*

Theorem 3.25 (Levy-Solovay)**.** *Suppose κ is measurable and $\mathbb{P} \in V_\kappa$ is a partial order. Then κ is measurable after forcing with $\mathbb{P}$.*

Proof. Let $j : V \to M$ witness that κ is measurable. Let $G \subseteq \mathbb{P}$ be generic over V. Since $j(\mathbb{P}) = \mathbb{P}$ and $M \subseteq V$, $G = j[G]$ is $j(\mathbb{P})$-generic over M. Thus j can be definably extended to $\hat{j} V[G] \to M[G]$, and thus κ is measurable in $V[G]$. □

Remark 3.26. *The above argument applies to many other types of large cardinals besides measurable.*

Lemma 3.27. *Suppose M is a transitive model of ZFC^-, κ is a regular cardinal in M, and $\mathbb{P} \in M$ is a partial order that M thinks is κ-closed. Suppose that in V, $M^{<\kappa} \subseteq M$ and $\mathcal{P}(\mathbb{P})^M \leq \kappa$. Then in V, there is a filter $G \subseteq \mathbb{P}$ that is generic over M.*

Proof. In V, enumerate the dense subsets of $\mathbb{P}$ that live in M as $\langle D_\alpha : \alpha < \kappa \rangle$. Build a descending sequence $\langle p_\alpha : \alpha < \kappa \rangle$ such that for all $\alpha < \kappa$, $p_{\alpha+1} \in D_\alpha$. The construction continues at limit stages $\lambda < \kappa$ because $\langle p_\alpha : \alpha < \lambda \rangle \in M$, and M thinks that there is a lower bound to the sequence. Let $G = \{q : \exists \alpha < \kappa (q \geq p_\alpha)\}$. Then G is $\mathbb{P}$-generic over M. $\square$

Theorem 3.28 (Kunen-Paris)**.** *Assume GCH. Suppose κ is measurable and $\mathcal{U}$ is a κ-complete normal ultrafilter on κ. Let $X \in \mathcal{P}(\kappa) \setminus \mathcal{U}$ be a set of regular cardinals. Let $F : X \to \kappa$ be a function satisfying the hypotheses of Easton's Theorem. Then there is a generic extension in which κ is measurable and $2^\alpha = F(\alpha)$ for all $\alpha \in X$.*

Proof. Let $\mathbb{P}$ be the Easton forcing defined according to F,

$$\mathbb{P} = \prod_{\alpha \in X}^{E} \mathrm{Add}(\alpha, F(\alpha)).$$

Note that

$$j(\mathbb{P}) \cong \mathbb{P} \times \prod_{\alpha \in j(X) \setminus \kappa}^{E} \mathrm{Add}(\alpha, F(\alpha)) := \mathbb{P} \times \mathbb{Q}.$$

For each $p \in \mathbb{P}$, $\mathrm{sprt}(j(p)) = \mathrm{sprt}(p)$, so in the above representation of $j(\mathbb{P})$, $j(p) = (p, 1_{\mathbb{Q}})$.

Since $\kappa \notin j(X)$, $\mathbb{Q}$ is κ^+-closed in M. By Lemma 3.20 and GCH, $|j(\kappa)| = |j(\mathbb{P})| = \kappa^+$. By Lemma 3.27, there is a filter $H \subseteq \mathbb{Q}$ which is generic over M.

Let $G \subseteq \mathbb{P}$ be generic over V. Then G is generic over $M[H] \subseteq V$, so $G \times H$ is $j(\mathbb{P})$-generic over M. Since $j[G] = G \times \{1_{\mathbb{Q}}\}$, the embedding can be extended to $\hat{j} : V[G] \to M[G \times H]$. Thus κ is measurable in $V[G]$. By Easton's Theorem, $V[G] \models \forall \alpha \in X (2^\alpha = F(\alpha))$. $\square$

Exercise 3.29. *Prove using similar techniques that if κ is measurable, then there is a forcing extension in which κ is measurable and GCH holds.*

3.4 I-support iterations

Suppose θ is an ordinal, $\xi < \theta$, and I is an ideal on $\theta \setminus \xi$ containing all singletons. We say that a partial order $\mathbb{P}$ is an *I-support iteration* if there is a sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \xi \leq \alpha \leq \theta, \xi \leq \beta < \theta \rangle$ such that:

1. For $\xi \leq \alpha \leq \theta$, $\mathbb{P}_\alpha$ is a collection of functions on $\alpha \setminus \xi$ with some partial order $\leq_\alpha$, and $\mathbb{P}_\theta = \mathbb{P}$.

2. For $\xi \leq \alpha < \theta$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a partial order, and $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, via the canonical map $p \mapsto (p \restriction \alpha, p(\alpha))$. The ordering $\leq_{\alpha+1}$ is defined via this isomorphism and the two-step iteration ordering.

3. For limit $\lambda \leq \theta$, $p \in \mathbb{P}_\lambda$ iff for all $\alpha \in [\xi, \lambda)$, $p(\alpha)$ is a $\mathbb{P}_\alpha$-name for an element of $\dot{\mathbb{Q}}_\alpha$ and $\{\alpha \in [\xi, \lambda) : p(\alpha) \neq \dot{1}_{\mathbb{Q}_\alpha}\} \in I$. We put $q \leq_\lambda p$ when $q \restriction \alpha \leq_\alpha p \restriction \alpha$ for all $\alpha \in [\xi, \lambda)$.

Note that the map $p \mapsto p \restriction \alpha$ is a projection from $\mathbb{P}_\theta$ to $\mathbb{P}_\alpha$ for $\xi \leq \alpha < \theta$, since if $p \in \mathbb{P}_\theta$, $q \in \mathbb{P}_\alpha$, and $q \leq p \restriction \alpha$, the $q \cup p \restriction (\theta \setminus \alpha)$ is an element of $\mathbb{P}_\theta$ below p.

Lemma 3.30. *Suppose κ is a regular cardinal, I is a κ-complete ideal on θ, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is an I-support iteration, and for each α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is $\check{\kappa}$-(directed-)closed. Then $\mathbb{P}_\theta$ is κ-(directed-)closed.*

Proof. Let $\delta < \kappa$ and let $\langle p_\alpha : \alpha < \delta \rangle$ be descending (or directed) in $\mathbb{P}_\theta$. Let $X = \bigcup_{\alpha<\delta} \operatorname{sprt}(p_\alpha) \in I$. We construct a lower bound q to the p_α's inductively. Since $\mathbb{Q}_0$ is κ-closed, there is a lower bound $q(0)$ to $\langle p_\alpha(0) : \alpha < \delta \rangle$. Suppose inductively that we have $q \restriction \alpha$ for all $\alpha < \beta \leq \theta$, and that $\operatorname{sprt}(q \restriction \alpha) \subseteq X$ for all $\alpha < \beta$. If β is a limit, then let $q \restriction \beta = \bigcup_{\alpha<\beta} q \restriction \alpha$. Then $\operatorname{sprt}(q \restriction \beta) \subseteq X$, and so $q \restriction \beta \in \mathbb{P}_\beta$, and it is a lower bound to all $p_\alpha \restriction \beta$, simply because $q \restriction \gamma \leq p_\alpha \restriction \gamma$ for $\gamma < \beta$ and $\alpha < \delta$. If $\beta = \alpha + 1$, then $q \restriction \alpha \Vdash \langle p_\gamma(\alpha) : \gamma < \delta \rangle$ is a descending (or directed) sequence in $\dot{\mathbb{Q}}_\alpha$. If $\alpha \notin X$, let $q(\alpha) = 1$, so that we maintain the hypothesis that $\operatorname{sprt}(q \restriction \beta) \subseteq X$. Otherwise, let $q(\alpha)$ be a name for a lower bound to $\langle p_\gamma(\alpha) : \gamma < \delta \rangle$. □

Lemma 3.31. *Suppose I is an ideal on θ, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is an I-support iteration, and $\xi < \theta$. Let $\dot{J}$ be a $\mathbb{P}_\xi$-name for the ideal generated by $I \restriction (\theta \setminus \xi)$ by closing under subsets. Then there is a $\mathbb{P}_\xi$-name for a $\dot{J}$-support iteration $\langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta : \xi \leq \alpha < \theta, \xi \leq \beta < \theta \rangle$ and a dense embedding from $\mathbb{P}_\theta$ to $\mathbb{P}_\xi * \dot{\mathbb{R}}_\theta$.*

Proof. As we learned in Section 1.5, forcing with $\mathbb{P}_\xi$ and then with $\dot{\mathbb{Q}}_\xi$ is equivalent to forcing with $\mathbb{P}_{\xi+1}$. Further, we have a translation $T_{\xi+1}$ of $\mathbb{P}_{\xi+1}$-names into $\mathbb{P}_\xi$-names for $\dot{\mathbb{Q}}_\xi$-names, yielding that $\mathbb{P}_{\xi+2} \cong \mathbb{P}_\xi * (\dot{\mathbb{Q}}_\xi * T_{\xi+1}(\dot{\mathbb{Q}}_{\xi+1}))$. For notational purposes, let T_ξ be the identity function, as no translation is required at that point. We want to propagate this situation inductively.

Suppose that $\eta \leq \theta$ and we have a $\mathbb{P}_\xi$-name for a J-support iteration $\langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta : \xi \leq \alpha < \eta, \xi \leq \beta < \eta \rangle$ along with translation functions $\langle T_\alpha : \xi \leq \alpha < \eta \rangle$. Suppose inductively that:

1. For $\xi \leq \alpha < \eta$, T_α translates $\mathbb{P}_\alpha$-names into $\mathbb{P}_\xi * \dot{\mathbb{R}}_\alpha$-names. In particular, for every σ which is a $\mathbb{P}_\xi$-name for an $\dot{\mathbb{R}}_\alpha$-name, there is a $\mathbb{P}_\alpha$-name τ such that $\Vdash_{\mathbb{P}_\xi} (\Vdash_{\dot{\mathbb{R}}_\alpha} \sigma = T_\alpha(\tau))$.

2. For each $\zeta < \eta$, the map $p \mapsto (p \restriction \xi, \langle T_\alpha(p(\alpha)) : \xi \leq \alpha < \zeta \rangle)$ is a dense embedding from $\mathbb{P}_\eta$ to $\mathbb{P}_\xi * \dot{\mathbb{R}}_\zeta$. Abbreviate this map by $p \mapsto (p_\ell, p_u)$.

3. For $\xi \leq \alpha < \eta$ and $p \in \mathbb{P}_\alpha$, $p \Vdash \varphi(\tau_1, \dots, \tau_n)$ iff

$$p_\ell \Vdash (p_u \Vdash \varphi(T_\alpha(\tau_1), \dots, T_\alpha(\tau_n))).$$

Suppose first that $\eta = \zeta + 1$. Then by induction, $\mathbb{P}_\eta \cong \mathbb{P}_\zeta * \dot{\mathbb{Q}}_\zeta \cong (\mathbb{P}_\xi * \dot{\mathbb{R}}_\zeta) * \dot{\mathbb{Q}}_\zeta$, via the map as in induction hypothesis (2). The translation function T_ζ and the fact that hypothesis (3) holds for it is given by induction. Extend the map to $\mathbb{P}_\eta$ by defining $\dot{\mathbb{S}}_\zeta = T_\zeta(\dot{\mathbb{Q}}_\zeta)$ and sending $p \mapsto (p \restriction \xi, \langle T_\alpha(p(\alpha)) : \xi \leq \alpha \leq \zeta \rangle)$. By (3), this map is an order-isomorphism into its range. It is a dense embedding because for every τ which is a $\mathbb{P}_\xi$-name for a $\dot{\mathbb{R}}_\zeta$-name for an element $\dot{\mathbb{S}}_\zeta$, there is a $\mathbb{P}_\zeta$-name σ such that $T_\zeta(\sigma)$ is forced to be equal to τ.

To carry forward the induction hypotheses, let T_η be the canonical translation function of $\mathbb{P}_\eta \cong (\mathbb{P}_\xi * \dot{\mathbb{R}}_\zeta) * \dot{\mathbb{Q}}_\zeta$-names into $\mathbb{P}_\xi$-names for $\dot{\mathbb{R}}_\zeta * \dot{\mathbb{S}}_\zeta \cong \dot{\mathbb{R}}_\eta$-names. Hypothesis (3) holds at $\alpha = \eta$ because for every generic $G \subseteq \mathbb{P}_\eta$ with $p \in G$, the dense embedding yields a generic $G_\ell * G_u \subseteq \mathbb{P}_\xi * \dot{\mathbb{R}}_\eta$ containing (p_ℓ, p_u) such that $V[G] = V[G_\ell][G_u]$, and for every $\mathbb{P}_\eta$-name τ, $\tau^G = (T_\eta(\tau)^{G_\ell})^{G_u}$.

Now suppose η is a limit ordinal. Let $\dot{\mathbb{R}}_\eta$ be a $\mathbb{P}_\xi$-name for the J-support iteration of the previous stages. Let $p \in \mathbb{P}_\xi$ and let $\dot{r}$ be a $\mathbb{P}_\xi$-name for an element of $\dot{\mathbb{R}}_\eta$. Let $p' \leq p$ and $X \in I$ be such that $p' \Vdash_\xi \operatorname{sprt}(\dot{r}) \subseteq \check{X}$. For each $\alpha \in X$, let $q(\alpha)$ be a $\mathbb{P}_\alpha$-name such that $T_\alpha(q(\alpha))$ is forced to be equal to $\dot{r}(\alpha)$. Defining q to be trivial outside X, we have that $q \in \mathbb{P}_\eta$ and for $\xi \leq \alpha < \eta$, $\Vdash_\xi (\Vdash_\alpha q_u(\alpha) = \dot{r}(\alpha))$. Thus $p' {}^\frown q$ maps below $(p, \dot{r})$, so the embedding is dense. The desired translation function T_η exists by the fact that $\mathbb{P}_\eta$ is densely embedded into $\mathbb{P}_\xi * \dot{\mathbb{R}}_\eta$. Hypothesis (3) continues to hold for the same reasons as in the successor case. □

Lemma 3.32. *Suppose κ is a regular cardinal, $\mathbb{P}$ is a κ-c.c. forcing, and I is a κ-complete ideal. Let $G \subseteq \mathbb{P}$ be generic, and let J be the ideal generated by I in $V[G]$. Then J is κ-complete in $V[G]$.*

Proof. Let $\delta < \kappa$ and let $\langle \dot{X}_\alpha^G : \alpha < \delta \rangle \subseteq J$. In V, for each $\alpha < \delta$, choose a maximal antichain $A_\alpha \subseteq \mathbb{P}_\xi$ such that for each $p \in A_\alpha$, there is $Y_\alpha^p \in I$ such that $p \Vdash \dot{X}_\alpha \subseteq \check{Y}_\alpha^p$. For each α, let $Z_\alpha = \bigcup_{p \in A_\alpha} Y_\alpha$. Then $Z_\alpha \in I$. Let $Z = \bigcup_{\alpha < \delta} Z_\alpha \in I$. Then $1 \Vdash \check{Z} \supseteq \bigcup_{\alpha < \delta} \dot{X}_\alpha$. □

Corollary 3.33. *Suppose κ is a regular cardinal, I is an ideal on θ, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is an I-support iteration, and $\xi < \theta$ is such that:*

1. *$\mathbb{P}_\xi$ is κ-c.c.*

2. *$I \restriction (\theta \setminus \xi)$ is κ-complete.*

3. *For $\alpha \geq \xi$, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is $\check{\kappa}$-(directed-)closed.*

Then it is forced that the quotient $\mathbb{P}_\theta/\mathbb{P}_\xi$ is equivalent to a κ-(directed-)closed poset.

Proof. Let $\dot{J}$ be a name for the ideal generated in $V^{\mathbb{P}_\xi}$ by I. Let $\langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta : \xi \leq \alpha < \theta, \xi \leq \beta < \theta \rangle$ be the $\mathbb{P}_\xi$-name for the $\dot{J}$-support quotient iteration given by Lemma 3.31. By the κ-c.c. and Lemma 3.32, $\dot{J}$ is forced to be κ-complete.

Now look at the proof of Lemma 3.31. By inductive hypothesis (3), each $\dot{\mathbb{S}}_\alpha$ is forced to be κ-(directed-)closed for $\xi \leq \alpha < \theta$. Thus by Lemma 3.30, $\dot{\mathbb{R}}_\theta$ is forced to be κ-(directed-)closed. □

3.5 Supercompact cardinals

A cardinal κ is λ-*supercompact* if there is a definable elementary embedding $j : V \to M$, where M is a transitive, λ-closed class, such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. We say κ is *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

Theorem 3.34. *The following are equivalent:*

1. *κ is λ-supercompact.*
2. *There is a normal, fine, κ-complete ultrafilter on*

$$\mathcal{P}_\kappa(\lambda) := \{z \subseteq \lambda : |z| < \kappa\}.$$

Proof. Suppose $j : V \to M$ witnesses that κ is λ-supercompact. Then $j[\lambda] \in M$. Let $\mathcal{U}$ be the collection of $A \subseteq \mathcal{P}_\kappa(\lambda)$ such that $j[\lambda] \in j(A)$. Since $\lambda < j(\kappa)$, we have that $\mathcal{P}_\kappa(\lambda) \in \mathcal{U}$. For all $z \in \mathcal{P}_\kappa(\lambda)$, $j[z] = z$, which has size $< \lambda$, so $\{z\} \notin \mathcal{U}$. Thus $\mathcal{U}$ is a nonprincipal ultrafilter.

$\mathcal{U}$ is κ-complete for the same reason as when we derive an ultrafilter on κ from j. $\mathcal{U}$ is fine because for any $\alpha < \lambda$, $j(\{z : \alpha \in z\}) = \{z : j(\alpha) \in z\} \ni j[\lambda]$. For normality, if $A \in \mathcal{U}$ and $f : A \to \lambda$ is regressive, then $j(f)(j[\lambda] = j(\alpha)$ for some particular $\alpha < \lambda$. Thus $\{z : f(z) = \alpha\} \in \mathcal{U}$.

Now suppose $\mathcal{U}$ is a normal, fine, κ-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. The ultrapower $V^{\mathcal{P}_\kappa(\lambda)}/\mathcal{U}$ is well-founded since $\mathcal{U}$ is countably complete, so it is isomorophic to a transitive class M. Let $j : V \to M$ be the derived map. The critical point of j is at least κ by κ-completeness, just as in the measurable case. Now by Łoś' Theorem $|[\mathrm{id}]| < j(\kappa)$, and by fineness, $j(\alpha) \in [\mathrm{id}]$ for all $\alpha < \lambda$. Thus $[\mathrm{id}]$ is a set of ordinals containing $j[\lambda]$, which M thinks is of size $< j(\kappa)$, so $\kappa = \mathrm{crit}(j)$ and $\lambda < j(\kappa)$. If $[f] \in [\mathrm{id}]$, then $\{z : f(z) \in z\} \in \mathcal{U}$, and thus by normality, f is constant on a set in $\mathcal{U}$, so $[f] = j(\alpha)$ for some $\alpha < \lambda$. Thus $[\mathrm{id}] = j[\lambda]$.

To show M is λ-closed, let $\langle x_\alpha : \alpha < \lambda \rangle \subseteq M$, and for each α, let f_α be a function of $\mathcal{P}_\kappa(\lambda)$ such that $x_\alpha = [f_\alpha] = j(f_\alpha)([\mathrm{id}])$. Let $\langle g_\alpha : \alpha < j(\lambda) \rangle = j(\langle f_\alpha : \alpha < \lambda)$. In M, take $\langle g_\alpha([\mathrm{id}]) : \alpha \in j[\lambda] \rangle$, which is a reindexed version of $\langle x_\alpha : \alpha < \lambda \rangle$. □

Exercise 3.35. *Show that if $\kappa \leq \lambda \leq \lambda'$ and κ is λ'-supercompact, then κ is λ-supercompact.*

Exercise 3.36. *Show that if κ is 2^κ-supercompact, then the set of measurable cardinals below κ is stationary.*

Exercise 3.37. *Show that κ is measurable iff κ is κ-supercompact.* Hint: *Show that if $\mathcal{U}$ is a normal, fine, κ-complete ultrafilter on κ, then $\kappa \in \mathcal{U}$.*

Theorem 3.38 (Solovay)**.** *If κ is λ-supercompact, where $\lambda \geq \kappa$ is regular, then $\lambda^{<\kappa} = \lambda$.*

Proof. Let $j : V \to M$ witness that κ is λ-supercompact. Since $M \models j(\lambda)$ is regular, $j[\lambda]$ is a bounded subset of $j(\lambda)$. Let $\gamma = \sup j[\lambda]$. Note that $j[\lambda]$ is a $<\kappa$-closed, unbounded subset of γ.

For each limit ordinal $\alpha < \lambda$, choose a club $C_\alpha \subseteq \alpha$ of ordertype $\operatorname{cf}(\alpha)$. Let $\langle D_\alpha : \alpha < j(\lambda)\rangle = j(\langle C_\alpha : \alpha < \lambda\rangle)$. It is easy to see that $D_\gamma \cap j[\lambda]$ is a $<\kappa$-closed, unbounded subset of γ. Let $C = j^{-1}[D_\gamma]$.

For $x \in \mathcal{P}_\kappa(C)$, $M \models j(x) = j[x] \subseteq D_\gamma$. Since $\operatorname{cf}(\gamma) = \lambda < j(\kappa)$, it follows by elementarity that $V \models \exists\alpha < \lambda(\operatorname{cf}(\alpha) < \kappa$ and $x \subseteq C_\alpha)$. Since κ is inaccessible, $|\mathcal{P}(C_\alpha)| < \kappa$ when $\operatorname{cf}(\alpha) < \kappa$. Thus $\lambda^{<\kappa} = |\mathcal{P}_\kappa(C)| \leq \kappa \cdot \lambda = \lambda$. □

Corollary 3.39. *If κ is supercompact and $\lambda > \kappa$ is singular of cofinality $< \kappa$, then $2^\lambda = \lambda^+$.*

Remark 3.40. *It is a theorem of Silver that if λ is singular of uncountable cofinality and $\{\alpha < \lambda : 2^\alpha = \alpha^+\}$ is stationary, then $2^\lambda = \lambda^+$. Using this and induction, we can remove the hypothesis that $\operatorname{cf}(\lambda) < \kappa$ in the above corollary.*

Lemma 3.41 (Laver)**.** *If κ is supercompact, then there is a function $f : \kappa \to V_\kappa$ such that for every $\lambda \geq \kappa$ and every $x \in H_{\lambda^+}$, there is a normal, fine, κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ such that $j_\mathcal{U}(f)(\kappa) = x$.*

Proof. Suppose this fails. Let $\varphi(g, \delta)$ be the statement that $\operatorname{dom} g$ is an inaccessible cardinal α, $g : \alpha \to V_\alpha$, and there is $x \in H_{\delta^+}$ such that there is no normal, fine, α-complete $\mathcal{U}$ on $\mathcal{P}_\alpha(\delta)$ with $j_\mathcal{U}(g)(\alpha) = x$. For each function $f : \kappa \to V_\kappa$, let λ_f witness $\varphi(f, \lambda_f)$. Let ν be greater than all λ_f, and let $j : V \to M$ witness that κ is ν-supercompact. Since $M^\nu \subseteq M$, M also satisfies $\varphi(f, \lambda_f)$ for each $f : \kappa \to V_\kappa$.

Now we construct a function $f : \kappa \to V_\kappa$ inductively. Given $f{\restriction}\alpha$, if $\varphi(f{\restriction}\alpha, \delta)$ holds for some $\delta < \kappa$, let x_α be a witness, and let $f(\alpha) = x_\alpha$. Otherwise, let $f(\alpha) = \emptyset$. Let $\lambda = \lambda_f$. By our supposition, $\varphi(f, \lambda)$ holds.

Let $x = j(f)(\kappa)$. Let $\mathcal{U} = \{A \subseteq \mathcal{P}_\kappa(\lambda) : j[\lambda_f] \in j(A)\}$. Let N be the ultrapower of V by $\mathcal{U}$, and let $k : N \to M$ be defined by $k([g]_\mathcal{U}) = j(g)(j[\lambda])$. Then k is elementary and $j = k \circ j_\mathcal{U}$.

Furthermore, for each $\alpha \leq \lambda$, $\alpha = \operatorname{ot}([\mathrm{id}]_\mathcal{U} \cap j(\alpha))$, so α is represented by the function $z \mapsto \operatorname{ot}(z \cap \alpha)$. Thus $k(\alpha) = \operatorname{ot}(j[\lambda] \cap j(\alpha)) = \alpha$. Therefore, $\operatorname{crit}(k) > \lambda$, and so $k(x) = x$. We have:

$$x = j(f)(\kappa) = k \circ j_\mathcal{U}(f)(k(\kappa)) = k\left(j_\mathcal{U}(f)(\kappa)\right).$$

Thus $x = j_\mathcal{U}(f)(\kappa)$, contradicting that x witnesses $\varphi(f, \lambda)$. □

Exercise 3.42. *Derive directly from the existence of such f as above (which are called* Laver functions*) that $\diamondsuit_\kappa$ holds. Recall that $\diamondsuit_\kappa$ states that there is a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ such that for all $X \subseteq \kappa$, $\{\alpha : X \cap \alpha = a_\alpha\}$ is stationary.*

3.6 Iterated forcing with supercompacts

Lemma 3.43. *Let $j : V \to M$ be a definable elementary embedding with critical point $\kappa \leq \lambda$, where M is a λ-closed transitive class. Suppose:*

1. *$\kappa \leq \theta \leq \lambda$, and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is an Easton-support iteration such that $|\mathbb{P}_\theta| \leq \lambda$.*
2. *For all $\alpha < \kappa$, $\mathbb{P}_\alpha \in V_\kappa$, and $\mathbb{P}_\theta = j(\mathbb{P}_\kappa) \restriction \theta$.*
3. *For all regular $\alpha < \theta$, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is α-directed-closed, and for nonregular α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \{1\}$.*
4. *For $\theta \leq \alpha \leq \lambda$, $M \models$ "$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \{1\}$."*

If $\eta \leq \lambda$ is a regular cardinal such that $2^\eta \leq \lambda^+$, then $\mathbb{P}_\theta$ forces that κ is η-supercompact.

Proof. First we argue that $\mathbb{P}_\kappa$ has the κ-c.c, by a Δ-system argument. Let $\langle p_\alpha : \alpha < \kappa \rangle \subseteq \mathbb{P}_\kappa$. For each regular α, there is $\beta_\alpha < \alpha$ such that $\text{sprt}(p_\alpha) \cap \alpha \subseteq \beta_\alpha$. Since κ is Mahlo, there is a stationary $S \subseteq \kappa$ of consisting of regular cardinals on which the function $\alpha \mapsto \beta_\alpha$ takes a constant value β. Since $\mathbb{P}_\beta \in V_\kappa$, there is a stationary $T \subseteq S$ such that $p_\alpha \restriction \beta$ takes a constant value q for $\alpha \in T$. Let C be the club of points α such that $\sup_{\gamma<\alpha}(\text{sprt}(p_\gamma)) \subseteq \alpha$. Then for $\alpha_0 < \alpha_1$ in $T \cap C$, p_{α_0} is compatible with p_{α_1}.

By Corollary 3.33, it is forced that the quotient $\mathbb{P}_\theta/\mathbb{P}_\kappa$ is κ-directed-closed. By assumption (4), $\mathbb{P}_{\lambda^+}$ is equivalent to $\mathbb{P}_\theta$, which has size $\leq \lambda$, and is thus λ^+-c.c. M believes that the Easton ideal restricted to $[\lambda, j(\kappa))$ is λ^+-complete. Thus if $G_0 \subseteq \mathbb{P}_\theta$ is generic, then $M[G_0]$ thinks that the quotient $j(\mathbb{P}_\kappa)/\mathbb{P}_\theta$ is λ^+-directed-closed. But this is true in $V[G_0]$ as well, since $M[G_0]^\lambda \cap V[G_0] \subseteq M[G_0]$. The reason for this is that any λ-sequence of ordinals in $V[G_0]$ is the evaluation of a $\mathbb{P}_\theta$-name of size λ, and this name is an element of M. Furthermore, if $G_1 \subseteq j(\mathbb{P}_\kappa)$ is any generic projecting to G_0, then $M[G_1]^\lambda \cap V[G_1] \subseteq M[G_1]$, because the forcing to get from G_0 to G_1 adds no λ-sequences of ordinals.

Let $\hat{G}$ be generic for $j(\mathbb{P}_\kappa)$, and decompose it as $G * H * K$, which is generic for $\mathbb{P}_\kappa * (\mathbb{P}_\theta/\mathbb{P}_\kappa) * (j(\mathbb{P}_\kappa)/\mathbb{P}_\theta)$. Since $j(p) = p ^\frown \vec{1}$ (where we append a tail of 1's in the appropriate forcing languages), we have that $j[G] \subseteq \hat{G}$. So by Silver's Lemma, the embedding can be extended to $j : V[G] \to M[\hat{G}]$.

Let $\mathbb{Q} = \mathbb{P}_\theta/G$. $M[\hat{G}]$ thinks that $j(\mathbb{Q})$ is $j(\kappa)$-directed-closed. Furthermore, $H \in M[\hat{G}]$, $j[H] \in M[G]$, and $j[H]$ is a directed subset of $j(\mathbb{Q})$, and of size $\leq \lambda < j(\kappa)$. Thus there is $q \leq j[H]$. If $\hat{H}$ is generic over $M[G]$ and contains q, then the embedding can be further extended to $j : V[G][H] \to M[\hat{G}][\hat{H}]$.

In $V[\hat{G}][\hat{H}]$, we may derive a normal, fine, κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\eta)$, with respect to the sets that exist in $V[G][H]$, by letting $\mathcal{U} = \{X \subseteq \mathcal{P}_\kappa(\eta) :$

$j[\eta] \in j(X)\}$. Since the quotient forcing $\mathbb{R} = j(\mathbb{P}_\kappa * \mathbb{Q})/(G * H)$ is λ^+-closed, this is an ultrafilter in the sense of $V[\hat{G}][\hat{H}]$ as well, so κ is η-supercompact in $V[\hat{G}][\hat{H}]$.

Let $\dot{\mathcal{U}}$ be an $\mathbb{R}$-name for this ultrafilter. Let $\langle X_\alpha : \alpha < 2^\eta \rangle$ enumerate all subsets of $\mathcal{P}_\kappa(\eta)$, and let $\langle f_\alpha : \alpha < 2^\eta \rangle$ enumerate all regressive functions on $\mathcal{P}_\kappa(\eta)$. Choose a descending sequence $\langle r_\alpha : \alpha < 2^\eta \rangle \subseteq \mathbb{R}$ such that r_α decides whether $X_\alpha \in \dot{\mathcal{U}}$ and decides some X_β to be a measure-one set on which f_α is constant. Let $\mathcal{U}^*$ be the filter generated by this coherent sequence of decisions. $\mathcal{U}^*$ witnesses that κ is η-supercompact in $V[G][H]$. □

Corollary 3.44. *Suppose κ is supercompact and $\xi \leq \kappa$. Then there is a generic extension in which κ is supercompact and $2^\kappa \geq \kappa^{+\xi}$.*

Proof. First suppose $\xi < \kappa$. Define a class-sized Easton-support iteration $\langle \mathbb{P}_\alpha : \alpha \leq \Omega \rangle$, where at inaccessible α, we force with $\mathrm{Add}(\alpha, \alpha^{+\xi})$, and do nothing at other points. Let $\lambda > \kappa$ be a regular cardinal such that $\lambda \geq \sup\{(\alpha^{+\xi})^{<\alpha} : \alpha < \lambda$ is inaccessible$\}$. Let $j : V \to M$ witness that κ is λ'-supercompact, where λ' is least such that $2^\lambda \leq (\lambda')^+$. Then $\mathbb{P}_\lambda = j(\mathbb{P}_\kappa) \restriction \lambda$, and $|\mathbb{P}_\lambda| \leq \lambda$. The other hypotheses of the previous lemma are satisfied, so κ is λ-supercompact after forcing with $\mathbb{P}_\lambda$. The forcing after $\mathbb{P}_\lambda$ doesn't add subsets of $\mathcal{P}_\kappa(\lambda)$, so κ-remains λ-supercompact after the full iteration.

For $\xi = \kappa$, the argument is similar except that at inaccessible α, we force with $\mathrm{Add}(\alpha, \alpha^{+\alpha+1})$. □

Exercise 3.45. *Show that if κ is supercompact, then there is a generic extension in which κ is supercompact and GCH holds.*

What about other possible behavior of the power function at κ? We need some way of anticipating behavior at κ by some choices below κ. This is where Laver functions come in, showing that anything consistent with the basic constraints of cardinal arithmetic can happen.

Theorem 3.46 (Laver)**.** *If κ is supercompact, then there is a κ-c.c. forcing $\mathbb{P}$ of size κ which forces that κ is supercompact and remains so after any κ-directed-closed forcing.*

Proof. Let $f : \kappa \to V_\kappa$ be a Laver function. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton-support iteration constructed inductively as follows. Suppose we have $\mathbb{P}_\alpha \in V_\kappa$ and an increasing sequence $\langle \lambda_\beta : \beta < \alpha \rangle \subseteq \kappa$. At stage α, let $\dot{\mathbb{Q}}_\alpha$ be the trivial partial order $\{1\}$ except in the case that:

1. α is inaccessible.
2. $\alpha > \lambda_\beta$ for all $\beta < \alpha$.
3. $f(\alpha)$ is a pair $(\dot{\mathbb{Q}}, \lambda)$, where $\alpha \leq \lambda < \kappa$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name for an α-directed-closed forcing of size $\leq \lambda$.

If these conditions obtain, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}$, and $\lambda_\alpha = \lambda$.

Now let $G \subseteq \mathbb{P}_\kappa$ be generic and let $\mathbb{Q}$ be any κ-directed-closed forcing in $V[G]$ (including possibly $\{1\}$). Let $\kappa \leq \eta \leq \lambda$ be regular such that $2^\eta \leq \lambda^+$ and $|\mathbb{Q}| \leq \lambda$. We may assume that $\mathbb{Q} \in H_{\lambda^+}$ and that it is the evaluation of a $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}} \in H_{\lambda^+}$. Since f is a Laver function, there is an ultrafilter $\mathcal{U}$ witnessing that κ is λ-supercompact, with $j(f)(\kappa) = (\dot{\mathbb{Q}}, \lambda)$, where $j = j_\mathcal{U}$.

Consider the iteration $j(\mathbb{P}_\kappa)$ with defining sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq j(\kappa), \beta < j(\kappa)\rangle$. Then $\dot{\mathbb{Q}}_\kappa = \dot{\mathbb{Q}}$ and $\lambda_\kappa = \lambda$. Thus for $\kappa < \alpha \leq \lambda$, $\dot{\mathbb{Q}}_\alpha = \{1\}$. Thus the hypotheses of Lemma 3.43 are satisfied, so κ is η-supercompact after forcing with $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$. □

A cardinal κ that is forced to be supercompact by every κ-directed-closed forcing is called *indestructibly supercompact.*

Exercise 3.47. *Show that if κ is indestructibly supercompact, then for every successor ordinal $\xi < \kappa$, the set $\{\alpha < \kappa : 2^\alpha = \alpha^{+\xi}\}$ is stationary.*

4 Generic large cardinals

4.1 General facts

Suppose I is an ideal over a set Z. Let us force with the Boolean algebra $\mathcal{P}(Z)/I$, obtaining a generic filter G. $\bigcup G$ is a subset of $\mathcal{P}(Z)^V$ disjoint from I; in fact it is an ultrafilter on the Boolean algebra $\mathcal{P}(Z)^V$ extending I^*. In $V[G]$, we can form the ultrapower of V, V^Z/G. It might be well-founded. In this case, we have an elementary embedding $j : V \to M \subseteq V[G]$, where M is a transitive subclass of $V[G]$. If V^Z/G is always well-founded, for any generic G, then we say that I is *precipitious.*

Lemma 4.1. *Suppose I is a precipitous ideal on Z, and $G \subseteq \mathcal{P}(Z)/I$ is generic. Then* $\mathrm{crit}(j_G)$ *is the unique κ such that I is κ-complete and for some $A \in G$, A is the union of κ-many sets from I.*

Proof. Let $B \in I^+$ be arbitrary. Let κ be such that $I \restriction B$ is κ-complete but not κ^+-complete. Then there is a positive $A \subseteq B$ which is the union of κ-many sets from I. Thus there is a dense set of conditions A such that $I \restriction A$ is κ-complete but A is partitioned into κ-many measure zero sets, so there is one such $A \in G$. If A, A' are two such sets and κ, κ' are their associated cardinals, then $A \cap A'$ is positive, $I \restriction (A \cap A')$ is κ-complete and κ'-complete but neither κ^+-complete nor $(\kappa')^+$-complete, so $\kappa = \kappa'$.

Let A be a set as above. We show by induction that $A \Vdash j(\alpha) = \alpha$ for $\alpha < \kappa$. Suppose this is true for $\beta < \alpha$. Let $B \subseteq A$ be positive and let $f : Z \to \alpha$. By κ-completeness, there is a positive $C \subseteq B$ on which f is constant. Thus $A \Vdash \exists \beta < \alpha([f]_{\dot{G}} = [c_\beta]_{\dot{G}})$. Thus $A \Vdash j(\alpha) = \alpha$.

Now let $\langle A_\alpha : \alpha < \kappa\rangle \subseteq I$ be a partition of A and consider the function $p : Z \to \kappa$ where $p(z) = \alpha$ iff $z \in A_\alpha$. Then $A \Vdash [f]_{\dot{G}} < j(\kappa)$, but also $[f]_{\dot{G}} > \alpha$ for each $\alpha < \kappa$. Thus $A \Vdash \mathrm{crit}(j) = \kappa$. □

Lemma 4.2. *Suppose I is a precipitous ideal on $Z \subseteq \lambda$. Then I is fine iff it is forced that $[\mathrm{id}]_G \supseteq j_G[\lambda]$, and I is normal iff it is forced that $[\mathrm{id}]_G \subseteq j_G[\lambda]$.*

Proof. If I is fine, then for each $\alpha < \lambda$, $\{z : \alpha \in z\} \in I^*$, so it is forced that $j(\alpha) \in [\mathrm{id}]$. If it is forced that $j(\alpha) \in [\mathrm{id}]$ for each $\alpha < \lambda$, then $\{z : \alpha \notin z\}$ must be in I.

Suppose that I is normal. Let A, f be such that $A \Vdash [f] \in [\mathrm{id}]$. For all positive $B \subseteq A$, there is a positive $C \subseteq B$ on which f is constant, so $A \Vdash [f] \in j[\lambda]$. Now suppose it is forced that $[\mathrm{id}] \subseteq j[\lambda]$. Let A be positive and let f be regressive on A. If G is generic with $A \in G$, then there is $B \in G$, a subset of A, such that $B \Vdash [f] = j(\alpha)$ for some particular $\alpha < \lambda$, so f takes value α at I-almost-all points in B. □

Theorem 4.3 (Foreman)**.** *Suppose I is a precipitous ideal on Z and $\mathbb{P}$ is a Boolean algebra. Let $j : V \to M \subseteq V[G]$ denote a generic ultrapower embedding arising from I. Suppose $\dot{K}$ is a $\mathcal{P}(Z)/I$-name for an ideal on $j(\mathbb{P})$ such that whenever $G * h$ is $\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}$-generic and $\hat{H} = \{p : [p]_K \in h\}$, we have:*

1. $1 \Vdash_{\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}} \hat{H}$ *is $j(\mathbb{P})$-generic over M,*
2. $1 \Vdash_{\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}} j^{-1}[\hat{H}]$ *is $\mathbb{P}$-generic over V, and*
3. *for all $p \in \mathbb{P}$, $1 \nVdash_{\mathcal{P}(Z)/I} j(p) \in \dot{K}$.*

Then there is $\mathbb{P}$-name $\dot{J}$ for an ideal on Z and a canonical isomorphism

$$\iota : \mathcal{B}(\mathbb{P} * \mathcal{P}(Z)/\dot{J}) \cong \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}).$$

Furthermore, $\dot{J}$ is forced to be precipitous and have the same completeness and normality and fineness properties of I.

Proof. Let $e : \mathbb{P} \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K})$ be defined by $p \mapsto ||j(p) \in \hat{H}||$. By (3), this map has trivial kernel. By elementarity, it is an order and antichain preserving map. If $A \subseteq \mathbb{P}$ is a maximal antichain, then it is forced that $j^{-1}[\hat{H}] \cap A \neq \emptyset$. Thus e is regular.

Whenever $H \subseteq \mathbb{P}$ is generic, there is a further forcing yielding a generic $G * h \subseteq \mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}$ such that $j[H] \subseteq \hat{H}$. Thus there is an embedding $\hat{j} : V[H] \to M[\hat{H}]$ extending j. In $V[H]$, let $J = \{A \subseteq Z : 1 \Vdash_{(\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K})/e[H]} [\mathrm{id}]_M \notin \hat{j}(A)\}$. In V, define a map $\iota : \mathbb{P} * \mathcal{P}(Z)/\dot{J} \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K})$ by $(p, \dot{A}) \mapsto e(p) \wedge ||[\mathrm{id}]_M \in \hat{j}(\dot{A})||$. It is easy to check that ι is order and antichain preserving.

We want to show the range of ι is dense. Let $(B, \dot{q}) \in \mathcal{P}(Z)/I * j(\mathbb{P})/\dot{K}$. Without loss of generality, there is some $f : Z \to V$ in V such that $B \Vdash \dot{q} = [[f]_M]_K$. By the regularity of e, let $p \in \mathbb{P}$ be such that for all $p' \leq p$, $e(p') \wedge (B, \dot{q}) \neq 0$. Let $\dot{A}$ be a $\mathbb{P}$-name such that $p \Vdash \dot{A} = \{z \in B : f(z) \in \dot{H}\}$, and $\neg p \Vdash \dot{A} = Z$. $1 \Vdash_{\mathbb{P}} \dot{A} \in J^+$ because for any $p' \leq p$, we can take a generic $G * h$ such that $e(p') \wedge (B, \dot{q}) \in G * h$. Here we have $[\mathrm{id}]_M \in j(B)$ and $[f]_M \in \hat{H}$, so $[\mathrm{id}]_M \in \hat{j}(A)$. Furthermore, $\iota(p, \dot{A})$ forces $B \in G$ and $q \in h$, showing ι is a dense embedding.

To show J is precipitous, take any generic $H * \bar{G} \subseteq \mathbb{P} * \mathcal{P}(Z)/\dot{J}$, and let $G * h = \iota[H * \bar{G}]$ and $\hat{H} = \{p : [p]_K \in h\}$. For $A \in J^+$, $A \in \bar{G}$ if and only if $[\mathrm{id}]_M \in \hat{j}(A)$. If $i : V[H] \to N = V[H]^Z/\bar{G}$ is the canonical ultrapower embedding, then there is an elementary embedding $k : N \to M[\hat{H}]$ given by $k([f]_N) = \hat{j}(f)([\mathrm{id}]_M)$, and $\hat{j} = k \circ i$. Thus N is well-founded.

If $f : Z \to \mathrm{Ord}$ is a function in V, then $k([f]_N) = j(f)([\mathrm{id}]_M) = [f]_M$. Thus k is surjective on ordinals, so it must be the identity, and $N = M[\hat{H}]$. Since $i = \hat{j}$ and $\hat{j}$ extends j, i and j have the same critical point, so the completeness of J is the same as that of I. Finally, since $[\mathrm{id}]_N = [\mathrm{id}]_M$ and $j \restriction \bigcup Z = \hat{j} \restriction \bigcup Z$, I is normal/fine in V if and only if J is normal/fine in $V[H]$. □

4.2 Collapsing

Suppose κ is a regular cardinal and $\lambda \geq \kappa$. $\mathrm{Col}(\kappa, \lambda)$ is the collection of partial functions defined on a bounded subset of κ, into λ. It is easy to see that this is κ-closed.

Lemma 4.4. *Suppose $G \subseteq \mathrm{Col}(\kappa, \lambda)$ is generic. Then $\bigcup G$ is a surjection from κ to λ.*

Proof. Let $p \in \mathbb{P}$ and $\alpha < \lambda$. There is $q \leq p$ such that $\alpha \in \operatorname{ran} q$. Thus for each $\alpha < \lambda$, the set of conditions forcing $\alpha \in \operatorname{ran} \bigcup \dot{G}$ is dense. □

Suppose $\mu < \kappa$ are regular cardinals. We define

$$\mathrm{Col}(\mu, {<}\kappa) := \prod_{\alpha<\kappa}^{<\mu-\mathrm{sprt}} \mathrm{Col}(\mu, \alpha).$$

Lemma 4.5. *Suppose $\mu < \kappa$ are regular and $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$. Then:*

1. $\mathrm{Col}(\mu, {<}\kappa)$ *is μ-closed and κ-c.c.*
2. $\mathrm{Col}(\mu, {<}\kappa)$ *forces $|\alpha| = \mu$ for all $\alpha \in [\mu, \kappa)$.*

Proof. For (1), μ-closure holds since each term in the product is μ-closed and the ideal of supports is μ-complete. The κ-c.c. follows from Δ-system argument, just like in the proof of Lemma 3.43. For (2), note that $\mathrm{Col}(\mu, {<}\kappa)$ projects to $\mathrm{Col}(\mu, \alpha)$ for each $\alpha < \kappa$. □

Lemma 4.6. *Suppose $\mathbb{P}$ is a κ-closed forcing and $\mathbb{Q}$ is a κ-distributive forcing that collapses $2^{|\mathbb{P}|}$ to κ. Then there is a complete embedding $e : \mathbb{P} \to \mathcal{B}(\mathbb{Q})$.*

Proof. Suppose H is generic for $\mathbb{Q}$. Note that since $\mathbb{Q}$ is κ-distributive, $\mathbb{P}$ is still κ-closed in $V[H]$. Let $p \in \mathbb{P}$. In $V[H]$, $\mathcal{P}(\mathbb{P})^V = \kappa$. Let $\{D_\alpha : \alpha < \kappa\}$ enumerate the dense subsets of $\mathbb{P}$ from V. Inductively build a descending chain

$\langle p_\alpha : \alpha < \kappa \rangle \subseteq \mathbb{P}$ such that $p_0 = p$ and $p_{\alpha+1} \in D_\alpha$ for all $\alpha < \kappa$. Thus G is $\mathbb{P}$-generic over V and $p \in G$.

Since $\mathbb{Q}$ collapses a cardinal $> \kappa$, it does not have the κ-c.c. Let $A \subseteq \mathcal{B}(\mathbb{Q})$ be a maximal antichain of size κ, and let us index it by $\mathbb{P}$: $A = \{a_p : p \in \mathbb{P}\}$. For each $p \in \mathbb{P}$, there is a $\mathbb{Q}$-name for a $\mathbb{P}$-generic filter $\dot{g}_p$ that is forced to contain p. Let $\dot{g}$ be a $\mathbb{Q}$-name such that for each $p \in \mathbb{P}$, $a_p \Vdash \dot{g} = \dot{g}_p$. What we have is that $\Vdash_{\mathbb{Q}} \dot{g}$ is $\mathbb{P}$-generic over V, and for each $p \in \mathbb{P}$, $||p \in \dot{g}|| > 0$. By Exercise 1.31, the map $p \mapsto ||p \in \dot{g}||$ is a complete embedding. □

4.3 Indestructibility of "ω_1 is generically supercompact"

We will say that a regular cardinal κ is *generically supercompact* if for cofinally many $\lambda \geq \kappa$, there is a κ-complete, normal, fine, precipitous ideal on $\mathcal{P}_\kappa(\lambda)$.

Lemma 4.7. *Suppose κ is generically supercompact and $\mathbb{P}$ is κ-c.c. Then $\mathbb{P}$ forces that κ is generically supercompact.*

Proof. Suppose there is a κ-complete normal fine precipitous ideal I on $\mathcal{P}_\kappa(\lambda)$. If $G \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)/I$ is generic, then any generic $\hat{H} \subseteq j(\mathbb{P})$ has the property that $j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over V, since $j(A) = j[A]$ for any maximal antichain $A \subseteq \mathbb{P}$ in V. Thus hypotheses of Foreman's Theorem are satisfied, with $\dot{K}$ as the trivial ideal on $j(\mathbb{P})$. □

From this we see that a successor cardinal such as ω_1 can be generically supercompact. If κ is supercompact and $\mu < \kappa$ is regular, then $\mathrm{Col}(\mu, {<}\kappa)$ forces that $\kappa = \mu^+$ and κ is generically supercompact.

Th generic supercompactness of ω_1 is automatically indestructible by countably closed forcing. This is different from the case of conventional supercompactness. The explanation is that the desired absorption property comes for free when we collapse sufficiently large cardinals, whereas in the conventional case, we must build the desired absorption property into the scheme of the iteration below κ.

Theorem 4.8. *If ω_1 is generically supercompact, then this is indestructible by countably closed forcing.*

Proof. Let $\mathbb{P}$ be any countably closed forcing. Let $\lambda \geq 2^{|\mathbb{P}|}$ be such that there is a normal fine precipitous ideal I on $\mathcal{P}_{\omega_1}(\lambda)$. We may assume that $\mathcal{P}(\mathbb{P})^V$ is coded into a subset $A \subseteq \lambda$. Let $j : V \to M \subseteq V[G]$ be a generic embedding arising from I. We have that $\mathcal{P}(\mathbb{P})^V \in M$, since it is decoded by the transitive collapse of $j(A) \cap j[\lambda]$. Since $2^{|\mathbb{P}|}$ is countable in M, we can build in M a filter $H \subseteq \mathbb{P}$ that is generic over V, containing any given $p \in \mathbb{P}$ we want. So there is a complete embedding $e : \mathbb{P} \to \mathcal{B}(\mathcal{P}(\mathcal{P}_{\omega_1}\lambda)/I)$ with the property that if G is generic for the latter and $H = e^{-1}[G]$, then $H \in M$.

In M, $j[H]$ is a countable directed subset of $j(\mathbb{P})$. By the countable closure of $j(\mathbb{P})$, we can find a $q \leq j[H]$. If we force below this, we get a filter $\hat{H}$ that is $j(\mathbb{P})$-generic over M, with $j[H] \subseteq \hat{H}$. Thus the hypotheses of Foreman's Theorem are satisfied, and there is a normal fine precipitous ideal J on $\mathcal{P}_{\omega_1}\lambda$ in $V[H]$. □